2. Stochastic processes

Stochastic process: collection of random variables $X(t)$

↳ Markov process: $P(x(t_i) \mid x(t_{i-1}), x(t_{i-2}), \dots) = P(x(t_i) \mid x(t_{i-1}))$

Goal: Find probability that X takes a value x at time t, $P(x,t)$

1) Consider specific realisations of $X(t)$

$$\dot{x} = f(x,t) + \xi(t)$$

$\Bigg\}$ Pool realisations of $X(t)$ to obtain $P(x,t)$

2) Directly consider time evolution of probability density $p(x,t)$

$$\frac{d}{dt} P(x,t) = F(P,x,t)$$

2.1 Langevin approach (stochastic differential equations)

General form of stochastic differential equation:

$$\boxed{\dot{x} = \underbrace{a(x,t)}_{\text{deterministic part}} + \underbrace{b(x,t)\,\xi(t)}_{\text{stochastic part}}}$$

Two classes of SDEs: 1) additive noise : $b = \text{const}$

2) multiplicative noise : $b = b(x)$

In differential form: $dx = a(x,t)\,dt + b(x,t)\,dW(t)$

$\hookrightarrow dW(t) = \xi(t)\,dt$

$W(t) = \int_0^t \xi(t)\,dt$

Formal solution:

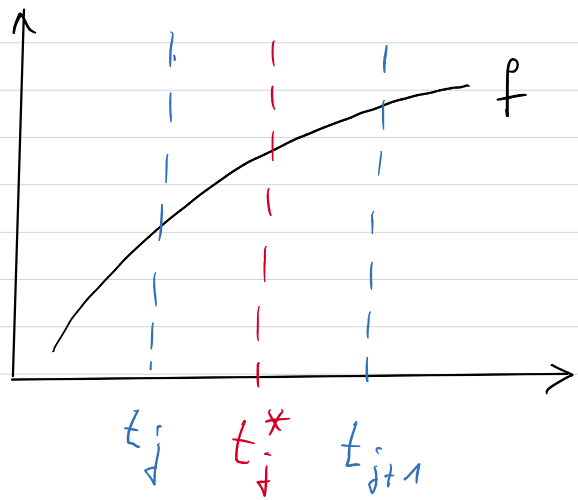
$$x(t) = x(0) + \int_{t_0}^{t} a(x,s)\,ds + \int_{t_0}^{t} b(x,s)\,dW_s$$

What is the second integral?

Partition Π_n of time domain into n intervals:

$$\int_{t_0}^{t} f(s)\,dW_s = \lim_{|\Pi_n| \to 0} \sum_{j=1}^{n} f(t_j^*)\left(W_{t_{j+1}} - W_{t_j}\right)$$

Stochastic integral is defined as weighted sum, where the weights are random variables.



The choice of t_j^* is important!

Example 1: left point $t_j^* = t_j$

$$\left\langle \sum_{j=1}^{n} W_{t_j} \left(W_{t_{j+1}} - W_{t_j} \right) \right\rangle$$

$$= \sum_{j=1}^{n} \left\langle W_{t_j} \left(W_{t_{j+1}} - W_{t_j} \right) \right\rangle$$

$$= \sum_{j=1}^{n} \langle W_{t_j} \rangle \langle W_{t_{j+1}} - W_{t_j} \rangle$$

$$= 0$$

Example 2: $t_j^* = t_{j+1}$ $\qquad f(t) = W_t$

$$\left\langle \sum_{j=1} W_{t_{j+1}} \left(W_{t_{j+1}} - W_{t_j} \right) \right\rangle$$

$$= \sum_{j=1}^{n} \left\langle \left(W_{t_{j+1}} - W_{t_j} \right)^2 \right\rangle$$

$$= \sum_{j=1}^{n} \left(t_{j+1} - t_j \right) = t - t_0$$

Two common choices are:

1) Ito integral: $t^* = t_j$ — this lecture

2) Stratonovich integral: $t_j^* = (t_{j+1} - t_j)/2 + t_j$

Chain rule in stochastic calculus:

$$df(x(t)) \overset{\text{Taylor}}{=} \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dx + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} dx^2 + \dots$$

Substitute LV for dx

$$= \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} (a\,dt + b\,dW) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \left(a^2 dt^2 + 2ab\,dt\,dW + b^2 dW^2\right) + \dots$$

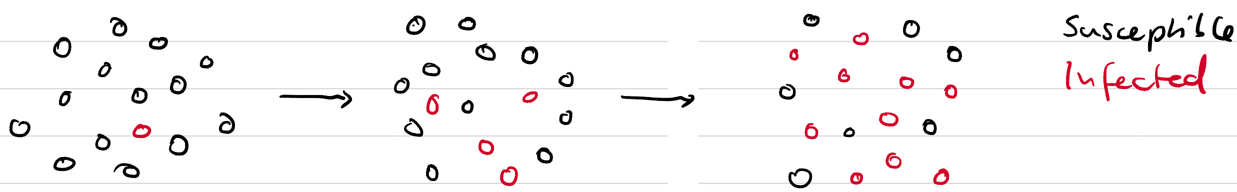
Use $dx(t) \propto \sqrt{dt}$ (SD of Brownian motion)

$$= \left(\frac{\partial f}{\partial t} + a \frac{\partial f}{\partial x} + \frac{b^2}{2} \frac{\partial^2 f}{\partial x^2}\right) dt + b \frac{\partial f}{\partial x} dW + \dots$$

Ito's formula

Example (Langevin)

Consider the unhindered spread of an epidemic.



Deterministically, the concentration of infected people follows $\dot{x} = a x$

The rate of spreading itself is subject to fluctuations, $a = \mu + \sigma \xi(t)$

Stochastic dynamics in differential form:

$$dx = \mu x \, dt + \sigma x \, dW(t)$$

Divide by x, $\frac{dx}{x} = \mu\, dt + \sigma\, dW(t)$

Apply Ito's formula with $f(x) = \log(x)$,

$$dy = d(\log(x)) = \left[\mu - \frac{\sigma^2}{2}\right] dt + \sigma\, dW$$

$$\frac{dx}{x} - \frac{\sigma^2}{2} dt$$

$$\Rightarrow \quad \frac{dx}{x} = d(\log x) + \frac{1}{2}\sigma^2 dt$$

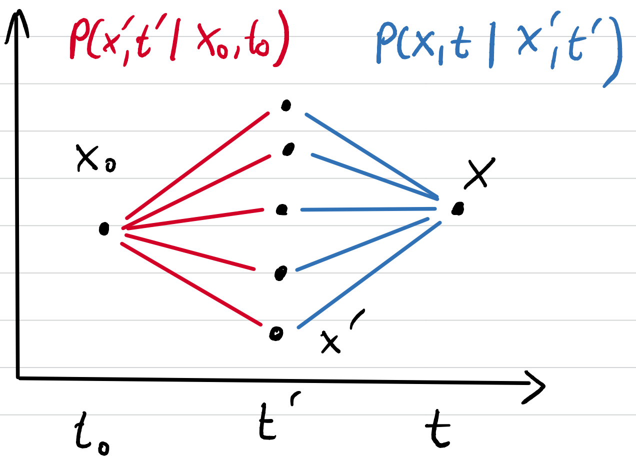
Integrating, and using that $\int_0^t \frac{dx}{x} = -\mu t - \sigma W(t)$, we get

$$\log \frac{x}{x_0} = \int_0^t \frac{dx}{x} - \frac{1}{2}\sigma^2 t = \mu t + \sigma W(t) - \frac{1}{2}\sigma^2 t$$

$$\Rightarrow \quad X(t) = x_0\, e^{(\mu - \frac{1}{2}\sigma)t + \sigma W_t}$$

3.2. Einstein approach

Probability of ending up in x at time t when starting at x_0 at t_0: $P(x,t \mid x_0,t_0)$



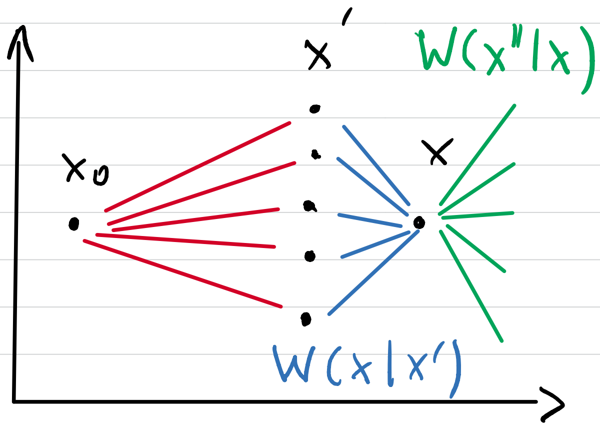
$P(x,t \mid x_0,t_0)$ is obtained by summing up contributions of different paths leading from x_0 to x:

$$\boxed{p(x,t \mid x_0,t_0) = \int p(x,t \mid x',t')\, p(x',t' \mid x_0,t_0)\, dx'}$$

Chapman - Kolmogorov equation

Goal: what is the time evolution of $p(x,t)$, $\frac{d}{dt} p(x,t) = ?$

Gain of probability in x is counteracted by loss to other states.



$$\boxed{\frac{d}{dt} p(x,t) = \underbrace{\int W(x|x')\, p(x',t)\, dx'}_{\text{flux into state } x} + \underbrace{\int W(x''|x)\, p(x,t)\, dx''}_{\text{flux out of state } x}}$$

Master equation

Example (Master equation)

Gene expression: Genes are read out by special molecules and transcribed into mRNA molecules. These are then translated to proteins which perform biological functions.

What is the probability that m mRNA molecules are produced in a time interval t?

$$\frac{d}{dt} p(m,t) = \lambda \left[p(m-1,t) - p(m,t) \right]$$

Introduce a characteristic function $G(s,t) = \langle e^{ims} \rangle = \sum_m P(m,t)\, e^{ims}$

Substituting into Master equation, $\partial_t G(s,t) = \lambda \left[e^{is} - 1 \right] G(s,t)$

$$\Rightarrow G(s,t) = e^{\lambda t [e^{is} - 1]} \Rightarrow P(m,t) = \frac{(\lambda t)^m}{m!} e^{-\lambda t}$$

Poisson distribution

Fokker Planck equation

Taylor expand around x in $\Delta x = x - x'$:

$$\partial_t P(x,t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \partial_x^n \alpha_n(x) \, p(x,t)$$

Kramers-Moyal expansion

with moments of the transition probabilities

$$\alpha_n(x) = \int d\Delta x \; W(x+\Delta x \,|\, x) \, \Delta x^n .$$

Truncate at second order

$$\boxed{\partial_t P(x,t) = \underbrace{- \partial_x \alpha_1(x) \, p(x,t)}_{\text{drift term: } \partial_t \langle x \rangle = \langle \alpha_1 \rangle} + \underbrace{\frac{1}{2} \partial_x^2 \alpha_2(x) \, p(x,t)}_{\text{diffusion term}}}$$

Fokker-Planck equation

Example (Fokker-Planck equation)

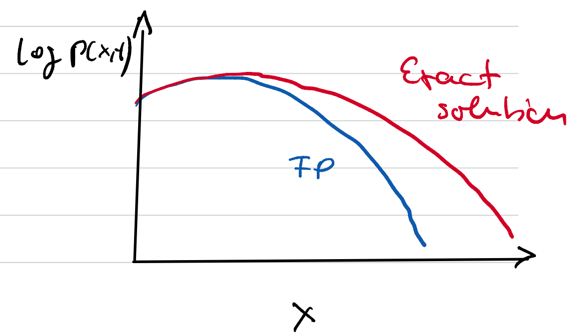
Fokker-Planck equation for the Poisson process.

Introduce rescaled variable $x = \frac{n}{N}$ with $p(x,t)\,dx = p(n,t)\,dn$.

Fokker-Planck approximation to the Master equation:

$$\partial_t\, p(x,t) = -\frac{\lambda}{N}\,\partial_x\, p(x,t) + \frac{\lambda}{2N^2}\,\partial_x^2\, p(x,t)$$

$$\Rightarrow \qquad p(x,t) = \frac{N}{\sqrt{2\pi\lambda t}}\, e^{-\frac{1}{2\lambda t}(Nx - t\lambda)^2}$$



FP and exact solution deviate in the tails.