METHODOLOGICAL NOTES

'Hermite' states in the quantum interaction of vortices

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Abstract. Quantum effects in the dynamics of a pair of monopolar vortices, arbitrary in their nature, are described in both the Heisenberg and Schrödinger languages. This system proves to be very closely and universally related to a linear quantum oscillator, which allows us to call it a 'Hermite' system, in regard to the eigenfunctions of such an oscillator.

1. Introduction

Two-dimensional point vortices are known to form highly curious mechanical systems. Their properties are far different from those of the habitual 'natural' systems, in which the kinetic and potential components of energy are clearly discriminated and which obey the normal, Newtonian laws of motion (see, e.g., Refs [1–4]). While the investigation of classical vortical dynamics has nevertheless been developed fairly well [2] (although highly surprising evolution patterns can be encountered even there [5–8]), quantum-mechanical approaches are still in their infancy (see Ref. [4] and the references therein) and do not yet go far beyond the determination of the energy spectra. (It is interesting that Hamiltonian spin dynamics, being no less specific than and somewhat similar to the quantum dynamics of vortices, is elaborated much better [9, 10].) For this reason, it appears important (even from a methodological standpoint) to advance in the quantum-mechanical techniques of description and to study not only energetic but also dynamical features of the quantum motion of vortices.

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2. Classical distributed vortices

First of all, we recall the basic properties of distributed vortical motions in continuous fluid media and the language used to describe such motions. A general, universal law, which is reflected by nearly all fundamental properties of ideal (dissipationless) flows, says that the curl of the generalized momentum of the fluid particles in a medium is frozen in the flow of this medium:

$$\frac{\partial \operatorname{rot} \mathbf{p}}{\partial t} = \nabla \times [\mathbf{v} \times \operatorname{rot} \mathbf{p}]. \tag{1}$$

This follows from the Hamiltonian nature of the motion of fluid particles and from the existence of the Poincaré – Cartan integral invariant in classical mechanics [2, 3]. The momentum $\mathbf{p}$ can be exemplified by the usual mechanical momentum $M\mathbf{v}$ in the dynamics of ideal fluids and the mechanical – field momentum $M\mathbf{v} + q\mathbf{A}/c$ in the magnetohydrodynamics of charged fluids (plasmas or electron pairs in a superconductor).

We here consider only two-dimensional flows of continuous media in the xy plane, and hence the curl of the generalized momentum is a (pseudo)scalar Π: $\nabla \times \mathbf{p} \equiv \Pi \mathbf{e}_z$. Because a (homogeneous) medium involved in vortical motion can, as a rule, be assumed to be incompressible, the velocity field is also specified by one (pseudo)scalar, the stream function $\mathbf{v} = \nabla \times (\Psi \mathbf{e}_z)$. All specific properties of various continuous fluid media (various types of fluid dynamics or various types of vortices) are determined by a linear relation between the two scalars:

$$\Pi = L(\Psi) \tag{2}$$

(for example, $L \propto \Delta$ in the dynamics of ideal fluids). An extremely important characteristic of the flow is the Green's function of Eqn (2), ψ, which allows an 'inversion' of the problem — the determination of the flow in terms of the given vorticity profile:

$$\Psi = \int \Pi(\mathbf{r}') \, \psi(\mathbf{r} - \mathbf{r}') \, \mathrm{d}^2 \mathbf{r}'$$

[obviously, $\psi = \psi(r)$ in an isotropic medium]. All key integrals of motion of a continuous medium can be expressed in the 'vorticity representation', precisely through this function. This transition from **v** as the base characteristic to $\nabla \times \mathbf{p}$ is equivalent (and similar in terms of convenience) to the transition from the field to charge representation in classical electrodynamics. Indeed, it can be readily seen that the energy, generalized momentum, and angular momentum of the fluid as a whole, which are conserved due to the original space–time symmetry of the system, can be rewritten as [5]

$$\mathcal{E} = \frac{nd}{2} \int \mathbf{p}\mathbf{v}\, \mathrm{d}^2\mathbf{r} = \frac{nd}{2} \int \Pi \varPsi \, \mathrm{d}^2\mathbf{r}$$

$$= \frac{nd}{2} \iint \Pi(\mathbf{r})\, \Pi(\mathbf{r}')\, \psi(\mathbf{r} - \mathbf{r}')\, \mathrm{d}^2\mathbf{r}'\, \mathrm{d}^2\mathbf{r}\,,$$

$$\mathbf{P} = nd \int \mathbf{p}\, \mathrm{d}^2\mathbf{r} = nd\, \mathbf{e}_z \times \int \mathbf{r}\Pi(\mathbf{r})\, \mathrm{d}^2\mathbf{r}\,, \qquad (3)$$

$$\mathbf{M} = nd \int \mathbf{r} \times \mathbf{p}\, \mathrm{d}^2\mathbf{r} = \frac{nd}{2}\, \mathbf{e}_z \int r^2 \Pi(\mathbf{r})\, \mathrm{d}^2\mathbf{r}\,,$$

where n is the concentration of fluid particles of the medium and d is the thickness of the medium in the z direction. We neglect the 'curvature' of the vortices; therefore, this thickness should be either very large ($d \to \infty$) compared to all other characteristic scales of the problem, for the 'end' effects to be negligible, or, conversely, very small ($d \to 0$), for any motion in the z direction to be absent in principle (up to dimensional quantization). We note in passing that not only does the form of ψ depend on the original relation between **p** and **v** but this function is also sensitive to the geometry of the problem (see below). However, only the first (energy) integral is specific to the given type of vortex, while the other two integrals are absolutely universal; the answers that emerge below are 'universal' precisely due to this fact.

3. Classical point vortices

It is conventional to attribute the term point vortices to local singularities of the vorticity field

$$\Pi = \sum_i \Gamma_i \delta(\mathbf{r} - \mathbf{r}_i)\,,$$

which are characterized by their intensities Γ_i and coordinates $\mathbf{r}_i$. Actually, vorticity is not concentrated at isolated points but is smeared over some area called the vortex core, of size a. If this quantity is small compared to other parameters of the problem (possibly, except d) — such as the distance between the closest vortices — the finite size of the core can be neglected, to a first approximation.

Here, we only consider vortices with the same amplitude (see the next section) and sign of intensity Γ_0. In this case, integrals (3) can be rewritten as

$$\mathcal{E} = N\Gamma_0^2 \sum_{i>j} \psi(\mathbf{r}_i - \mathbf{r}_j)\,,$$

$$\mathbf{P} = N\Gamma_0\, \mathbf{e}_z \times \sum_i \mathbf{r}_i\,,$$

$$M = \frac{N\Gamma_0}{2} \sum_i r_i^2\,.$$

Here, we use the notation $N \equiv nd$ for the linear (two-dimensional) concentration of the medium; the energy $\mathcal{E}$ is renormalized by eliminating the self-energy of the vortices (infinite at $a = 0$), which has no effect on their relative motion. (We can again draw an analogy to charges in classical electrodynamics: it is convenient to study their mechanical motion assuming that they are points and eliminating their self-energy.) The second integral, which expresses the conservation of the 'center of gravity' for a vortical system of identical 'particles', is trivial and has no significant effect on the dynamics of their relative motion.

It can be easily seen that according to fundamental equation (1), each vortex in such a discrete system is carried by the flow produced by all other vortices, with no variation in its intensity, and therefore the problem of the evolution of the continuous medium (formulated in terms of partial differential equations) reduces to a mechanical problem of the motion of individual 'particles' (described by ordinary differential equations). It is this fact that makes such a Π representation so attractive (in addition, it is warranted by genuine physical, quantum effects — see Ref. [4] and our presentation below). As is well known, this mechanical problem is Hamiltonian, with $\Gamma_0^2 \sum_{i>j} \psi(\mathbf{r}_i - \mathbf{r}_j)$ playing the role of the Hamiltonian function, $x_i(t)$ and $y_i(t)$ playing the role of canonical variables, and the corresponding Poisson brackets given by the formula [1–4]

$$\{f, g\} \equiv \frac{1}{\Gamma_0} \sum_i \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} \right).$$

These properties determine the following features of vortical dynamics. If we focus our attention on the dynamics in the xy plane, we see that in sharp contrast to Newtonian mechanics, the positions of particles determine their velocities rather than accelerations (on this basis, Kozlov [3] proposed to call the mechanics of point vortices Cartesian); this means that monopole vortices have no inertia, their motion is only controlled by 'particle' interaction, and they cannot be characterized by any mass. If, however, we note that the configuration space of the vortices is also their phase space, such that one normal coordinate (for example, x) can be regarded as a generalized coordinate and the other (y, according to our choice) as a generalized momentum (which halves the 'effective' dimensionality), we observe that the Hamiltonian of the problem, H, can in no way be represented as the simple sum of a potential energy $U(x)$ and a kinetic energy $\propto y^2$. This unnatural (in the strict meaning of this word) situation results in numerous unexpected peculiarities in the behavior of discrete vortical system [6–8].

Because quantum mechanics appears to be technically much more sophisticated than classical mechanics, we here consider only the pair interaction of vortices. Upon passing to the variables $\mathbf{R} = \mathbf{r}_1 + \mathbf{r}_2$ and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, the classical dynamics of such a system proves to be controlled by the integrals of motion

$$\mathcal{E} = N\Gamma_0^2 \psi(r)\,, \qquad \mathbf{P} = N\Gamma_0\, \mathbf{e}_z \times \mathbf{R}\,, \qquad M = \frac{N\Gamma_0}{4} r^2 + \mathrm{const}\,,$$

which specify the laws $\mathbf{R} = \mathrm{const}$ and $r = \mathrm{const}$. If we introduce new canonical variables q and p expressed through the components of the vector $\mathbf{r}$ as $q = \sqrt{N\Gamma_0/2}\, x$ and $p = \sqrt{N\Gamma_0/2}\, y$ (not to be confused with the above-mentioned canonical momentum of fluid particles), the classical (effectively 'one-dimensional') equations of motion assume

the form

$$
\begin{cases}
\dot{q} = \{q, H\}, \\
\dot{p} = \{p, H\},
\end{cases}
\qquad
H = N\Gamma_0^2 \psi\left(\sqrt{q^2 + p^2}\right), \quad M = \frac{q^2 + p^2}{2}
$$

$$(4)$$

with the usual Poisson bracket in this case. Here, along with the specific Hamiltonian, we intentionally write the 'universalizing' momentum integral (for a system of two vortices opposite in sign, the momentum integral plays a similar role [4, 5]; however, this case is much more similar to that of the motion of ordinary particles, because the dipole vortex has 'inertia', and we do not consider it here). The quantization of such a system is ideologically trivial [4].

4. Quantizing the intensities and interaction energy of vortices

The first implication of the quantum-mechanical effects is the discretization of the possible vorticity values for an individual vortex Γ_0. Indeed, according to Bohr's rule, an integer number of de Broglie waves must fit into any closed contour encompassing the vortex core, i.e.,

$$
\oint \mathbf{p}\, d\mathbf{r} = \int \Pi\, d^2\mathbf{r} = 2\pi\hbar \cdot (\pm 1, \pm 2, \dots).
$$

Because vortices with multiple 'charges' are unstable and decompose into elementary vortices, we can assume in our case that $\Gamma_0 = 2\pi\hbar$ (the choice of a unique sign is not important in principle and affects only the rotational direction). Linguistically, the standard quantum-mechanical derivation of this discreteness (see, e.g., Ref. [4]) deals with the wave-function phase rather than vorticity, but it coincides in essence with the above derivation.

As regards equations of motion (4), they can easily be quantized by replacing the Poisson bracket of functions, $\{f, g\}_{q, p}$, with the commutator of operators, $[\hat{f}, \hat{g}] \equiv \hat{f}\hat{g} - \hat{g}\hat{f}$, and taking the relation $[\hat{q}, \hat{p}] = i\hbar$ into account. It can easily be seen that the energy spectrum of a system of two identical vortices can then be obtained from the Bohr – Sommerfeld rule (in Ref. [4] and the references therein, this procedure is called 'geometric quantization'), according to the formula

$$
\oint p\, dq = \pi(q^2 + p^2) = 2\pi\hbar\left(k + \frac{1}{2}\right). \tag{5}
$$

Here, the integration in the phase plane (and, simultaneously, in a regular plane!) is performed along the trajectory of motion of the classical system — a circle of radius $\sqrt{q^2 + p^2}$. From here on, k is the principal quantum number. We note the following interesting circumstance (which was unfortunately ignored by Blatter et al. in their excellent review article [4]). In fact, the angular momentum of the vortical system,

$$
M = \hbar\left(k + \frac{1}{2}\right),
$$

is quantized here; due to the Hamiltonian 'one-dimensionality' of the system, it contains $1/2$ instead of 1, which is typical of usual motions in a plane (cf. Ref. [11]). It can readily be seen that the obtained spectrum $H(\sqrt{k + 1/2})$ corresponds in its physical meaning to especially highly excited,

extremely semiclassical energy levels, because, according to Eqns (4),

$$
k \sim \frac{\pi N r^2}{2} \gg 1, \tag{6}
$$

where r is the physical distance between the two vortices; evidently, this distance far exceeds $N^{-1/2}$ in the continuous-medium approximation. Undoubtedly, such a large dimensionless factor results from differences between the quantization of the motion of individual particles and the quantization of collective flows of a continuous medium.

Apparently, it is due to this circumstance that no interest has been given in the literature to the quantum laws of motion in vortical systems. From the practical standpoint, it is quite sufficient to restrict the consideration to the classical mechanical equations for a system of point vortices, taking only the discreteness of their intensities into account (similarly, a classical description of the motion of charged particles is possible in many problems of plasma physics, although the internal structure of these particles is exclusively controlled by quantum physics).

Nevertheless, it seems ideologically very important to consider the specific features of the quantum dynamics of such nontrivial Hamiltonian systems, which is manifested even in the energy spectra $\mathcal{E}_k$ obtained previously in many studies. As physical examples of 'unnatural' vortical Hamiltonians $H \propto \psi$, we can mention $-\ln r$ for an ideal (or superfluid) liquid [1–3]; the Macdonald function $K_0(r/b)$ for plasmas and massive superconductors (here, $b = c/\omega_{\mathrm{pe}}$ is the London length, or collisionless-skin length) [4, 5, 12]; a combination of the Struve and Neumann functions, $\mathbf{H}_0(r/b') - N_0(r/b')$, for Pearl vortices in superconducting films and current sheets in plasmas ($b' = c^2/(\omega_{\mathrm{pe}}^2 d)$) [4, 5, 12–14], which changes into $1/r$ for $r \gg b'$; or $1/r^4$, which characterizes the long-range fluctuational (Van der Waals) interaction of vortices in layered superconductors [15, 16]. It is most interesting that the answers prove to be highly universal and apply simultaneously to the above-mentioned functions and to any other symmetric functions $\psi(r)$.

5. Heisenberg representation

Most closely related to the classical technique of describing dynamical systems (4) is Heisenberg's ideology of discretization of continuous variables. Although it has remained nearly unclaimed in quantum mechanics by virtue of severe technical difficulties emerging in practical problems [11], it unexpectedly proves to be an extremely convenient and adequate investigation tool in our case of nontrivial vortical Hamiltonians.

According to this approach, the transition from the canonical variables q, p, and H to infinite matrices should be made in Eqns (4), as noted above, with replacing the Poisson bracket by the commutator. In view of the commutation relation for canonical variables, $[\mathsf{q}, \mathsf{p}] = i\hbar\mathsf{E}$ (where E is the unit matrix), this yields

$$
\begin{cases}
\dot{\mathsf{q}} = \dfrac{1}{i\hbar}\,[\mathsf{q}, \mathsf{H}], \\[2mm]
\dot{\mathsf{p}} = \dfrac{1}{i\hbar}\,[\mathsf{p}, \mathsf{H}],
\end{cases}
\qquad
\mathsf{H} = N\Gamma_0^2 \psi\left(\sqrt{\mathsf{q}^2 + \mathsf{p}^2}\right). \tag{7}
$$

The dynamical problem is considered solved if it is possible to find q and p satisfying Eqns (7) such that H proves to be a

diagonal matrix and the kth number in its diagonal is (the above-determined) $\mathcal{E}_k$.

Although the classical problem is essentially nonlinear, the quantum-mechanical answer is virtually identical to that found by Heisenberg himself for a (one-dimensional) harmonic (i.e., linear) oscillator: because the combination $\mathsf{q}^2 + \mathsf{p}^2$ (angular momentum M!) is diagonal in this case, the nonlinear function $\mathsf{H}(\sqrt{\mathsf{q}^2 + \mathsf{p}^2})$ is trivial:

$$\mathsf{q} = \mathrm{i}\sqrt{\frac{\hbar}{2}}\begin{pmatrix} 0 & 1 & 0 & \cdots \\ -1 & 0 & \sqrt{2} & \cdots \\ 0 & -\sqrt{2} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

$$\mathsf{p} = \sqrt{\frac{\hbar}{2}}\begin{pmatrix} 0 & 1 & 0 & \cdots \\ 1 & 0 & \sqrt{2} & \cdots \\ 0 & \sqrt{2} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \quad (8)$$

$$\mathsf{H} = N\Gamma_0^2\begin{pmatrix} \psi(1) & 0 & 0 & \cdots \\ 0 & \psi(\sqrt{3}) & 0 & \cdots \\ 0 & 0 & \psi(\sqrt{5}) & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

For brevity, we have written only the amplitude values of the matrices. We recall that the phase of each element A_{ij} of the matrix A is, generally, $\exp\left[\mathrm{i}(\mathcal{E}_i - \mathcal{E}_j)t/\hbar\right]$ [11]. Because $\psi(r)$ in all physical problems is a monotonically decreasing function of its argument (see Refs [5, 7]), the eigenenergy $\mathcal{E}_k$ of a vortical pair decreases as k increases.

6. Schrödinger representation

We can now consider a sort of projection of the actual two-dimensional motion in the xy plane onto the q axis and find what the Schrödinger approach to the description of quantum vortical motion yields.

Obviously, we must consider the procedure for solving the familiar equation

$$\mathrm{i}\hbar\,\frac{\partial\varphi}{\partial t} = \hat{H}\left(\sqrt{q^2 - \hbar^2\,\frac{\mathrm{d}^2}{\mathrm{d}q^2}}\right)\varphi \quad (9)$$

(we here let the wave function be denoted by φ to avoid confusion with ψ used above). Generally, imparting a clear mathematical meaning to nonlocal operators (such as the square root or even the logarithm of a derivative) is not a trivial task (see, e.g., Ref. [17]). However, it can be resolved quite simply in this case. We can use the completeness of the eigenfunction system for a one-dimensional (in q) harmonic oscillator to represent the sought wave function φ as

$$\varphi = \sum_k c_k \varphi_k\,,$$

where

$$\varphi_k = \frac{1}{\sqrt[4]{\hbar}\,\sqrt{2^k k!}}\exp\left(-\frac{q^2}{2\hbar}\right)H_k\left(\frac{q}{\sqrt{\hbar}}\right),$$

with H_k being the Hermite polynomials. Then, because

$$\left(\sqrt{q^2 - \hbar^2\,\frac{\mathrm{d}^2}{\mathrm{d}q^2}}\right)\varphi_k = \sqrt{2k+1}\,\hbar\varphi_k\,,$$

we obviously have

$$\hat{H}\varphi = \sum_k c_k\mathcal{E}_k\varphi_k\,.$$

Therefore, the eigenfunctions of the Hamiltonian for the pair interaction of vortices are the eigenfunctions (cf. the preceding section) of the operator $\hat{M}$, which coincide with the standard Hermite functions. This answer is completely universal [which, as promised, is an obvious manifestation of the universal nature of law (1)] and applicable to any kind of vortices in any isotropic medium. Differences between hydrodynamic behaviors manifest themselves only in the spectrum $\mathcal{E}_k \propto \psi(2k+1)$ and the time dependence of φ_k. However, these differences may be important; for example, because the energy levels are not equidistant here, no coherent states typical of a normal oscillator are present [11].

It is interesting that the above solution is exact rather than semiclassical. We can conclude that physically highly excited levels of the quantum motion of (any) two interacting monopolar vortices can be described (in quantum physics) by the classical Hermite functions (cf. the preceding section). For this reason, we find it expedient to introduce the term Hermite states by analogy with the currently popular Rydberg states in highly excited atoms and ions [18]. We emphasize once again that this property of reducibility to the basis model applies rigorously in vortical dynamics.

7. Anisotropic vortices

The isotropy of the original fluid medium, which gives rise to a unified integral of motion M, additional to the specific energy, is in no way a necessary attribute of the vortical problem. Another example of a nontrivial Hamiltonian is the problem of vortical motion in an HTSC-ceramics-type layered superconducting medium, in which the electron flow is only permitted in the xy plane [15]. At distances that are long compared to the London length, the interaction of two vortices that are identically tilted (i.e., parallel to the line $x = 0$, $y - z\tan\alpha = 0$) obeys the law

$$\psi \propto \frac{\tan^2\alpha(x^2 - y^2)}{(x^2 + y^2)^2}\,,$$

and hence their classical motion (for instance, in the plane $z = 0$) is described by Eqns (4) with the Hamiltonian

$$H = \frac{A}{2}\left[(q^2 - p^2)\,\frac{1}{(q^2 + p^2)^2} + \frac{1}{(q^2 + p^2)^2}(q^2 - p^2)\right], \quad (10)$$

where A is a proportionality factor. We symmetrized the Hamiltonian from the very beginning, keeping in mind the subsequent transition to a quantum-mechanical description, which implies that the operators constituting this Hamiltonian are not commutative. Actually, the level contours of ψ near the coordinate origin (at $r \leqslant c/\omega_{\mathrm{pe}}$) are closed ellipses (see Ref. [15]), which deviate from the contours given by Eqn (10) in an understandable way; however, this effect can be taken into account in perturbation theory for the system in Eqns (4) and (10).

The Bohr–Sommerfeld quantization of the energy levels of motion (5), taking into consideration that the classical trajectory of motion specified by the condition $H = \mathcal{E}$ is a Bernoulli lemniscate (whose lobes are isolated from each

other, according to the above remark on the reconnection of the level contours of H), yields the spectrum (relative to the lemniscate area; see Ref. [19])

$$\mathcal{E}_k = \frac{A}{2\pi\hbar(2k+1)} \, ; \tag{11}$$

here, negative k values, which describe motion along the lobes turned by $90°$, are quite admissible. Actually, the quantum 'tunnel' interaction of the lobes that are separated in the case of classical motion should result in the splitting of energy levels (into odd and even states) and in their small shift with respect to those given by Eqn (11); this effect, however, can also be included in perturbation theory.

Unfortunately, the use of the matrices q and p from Eqns (8) in the Heisenberg representation is not advantageous in this case, because H proves to be a two-diagonal, rather than diagonal, matrix: it has nonvanishing components H_{ij} with $i = j \pm 2$. However, Born, Heisenberg, and Jordan have shown in their classic study [20] that if we can diagonalize this matrix using the transformation $\mathsf{H}_0 = \mathsf{S}^{-1}\mathsf{HS}$ with a matrix S, then we obtain $\mathsf{q}_0 = \mathsf{S}^{-1}\mathsf{qS}$ and $\mathsf{p}_0 = \mathsf{S}^{-1}\mathsf{pS}$, respectively (because this transformation preserves the commutation relation between the canonical variables). Because we already know the eigenvalues H_0, we also know the matrix $\mathsf{H} - \lambda_k \mathsf{E}$, $k = 1, 2, \ldots$, which can be used to obtain the eigenvectors of the transformation and to subsequently construct S. In other words, the new problem also offers at least a recursive relation for the construction of its Heisenberg description.

Implementation of the Schrödinger ideology also involves some difficulties. Of course, the expansion of the eigenvalues φ_k of $\hat{H}$ in terms of the Hermite functions makes it possible to remove the annoying denominator in Eqn (10), but the numerator again permits us to obtain only a recursive relation for the expansion coefficients c_i^k (here, k is the index of an eigenfunction rather than a power, and i is the current index rather than the imaginary unit):

$$\frac{c_i^k}{\pi(2k+1)} = c_{i+2}^k (i+2)(i+1)\left[\frac{1}{(2i+5)^2} + \frac{1}{(2i+1)^2}\right]$$
$$+ \frac{c_{i-2}^k}{2}\left[\frac{1}{(2i+1)^2} + \frac{1}{(2i-1)^2}\right].$$

It can be seen that the odd and even Hermite functions appear independently in the expansion (this is also evident from the symmetry of the problem; see above). The solution of this recursion is not known to us, and we therefore write an explicit expression only for the function with $k = \infty$ ($\mathcal{E} = 0$), which corresponds to the 'separatory' (along $x = \pm y$) motion in the lemniscate. This function is a solution of the equation

$$\hbar^2 \frac{\mathrm{d}^2\varphi}{\mathrm{d}q^2} + q^2\varphi = 0\,,$$

i.e.,

$$\varphi_\infty \propto \sqrt[4]{\frac{q^2}{\hbar}}\, J_{\pm 1/4}\left(\frac{q^2}{2\hbar}\right),$$

where $J_{\pm 1/4}$ is the Bessel function (two solutions are present due to the degeneracy of the considered infinite motion and the merger of motions with $k > 0$ and $k < 0$ at the separatrix).

In complete agreement with the oscillation theorem [11], such a φ function has an infinite number of zeros.

Nevertheless, although the anisotropic case is far different from the original situation, it allows us to make significant progress in studying two-dimensional vortical systems based on the standard solution of the problem of a one-dimensional harmonic oscillator, in the framework of both the Heisenberg and Schrödinger approaches.

8. Conclusion

We have demonstrated that the quantum dynamics of a vortical system is, in some respects, highly universal and unified, even for media substantially differing in their hydrodynamics (ideal fluids, plasmas, superconductors, etc.). Based on the language developed to describe a normal linear oscillator, it can be characterized as 'Hermite' (in terms of its close relation to the Hermite functions rather than self-adjointness). The nature of this relationship is in the coincidence of the additional momentum integral in two-dimensional, 'Cartesian' vortical motion with the Hamiltonian of a one-dimensional, 'Newtonian' system.

A practical observation of the examined quantum dynamics appears to be a very difficult but not hopeless task. Obviously, thin films of superfluid helium would be the most appropriate object of observation. Indeed, because both d and a can (at least, in principle) be of the order of atomic sizes for such films, the necessary condition $r > a$ does not contradict the realization of a regime with $k \sim 1$ [see condition (6)], i.e., it does not contradict a strong 'shot' effect in the possible distances r. However, atomic sizes and energies must also be resolved. Potentially, similarly thin superconducting films have macroscopic a values (the coherence lengths, or the sizes of the Cooper pairs), which are in no way less than 100 Å; therefore, $k > 10^4$ for them.

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