Plasma Physics Reports, Vol. 30, No. 3, 2004, pp. 214–217. Translated from Fizika Plazmy, Vol. 30, No. 3, 2004, pp. 241–244.

Nonlinear Dynamics of Electron Vortex Lattices

V. Yu. Zaburdaev, V. V. Smirnov, and K. V. Chukbar

Russian Research Centre Kurchatov Institute, pl. Kurchatova 1, Moscow, 123182 Russia

Received April 14, 2003; in final form, July 24, 2003

Abstract—Weak and strong nonlinearities that determine the evolution of regular ensembles of electron vortices in a magnetized plasma are analyzed. Qualitative differences in behavior between such a medium and standard nonlinear media are revealed. *© 2004 MAIK "Nauka/Interperiodica".*

In recent years, it has become rather popular to represent magnetized plasma as a medium filled with two-dimensional vortices, vortex filaments, or other vortex structures (see, e.g., [1]). Therefore, investigation into the dynamics of large chaotic or regular vortex ensembles seems to be a very interesting and challenging problem. In [2], the following equation describing the evolution of long-wavelength nonlinear perturbations in a triangular lattice (the only one that is stable) of identical electron-type two-dimensional point vortices was derived:

$$\frac{\partial \boldsymbol{\xi}}{\partial t} = -R\mathbf{e}_z \times \boldsymbol{\nabla}(\nabla \cdot \boldsymbol{\xi}) + D\boldsymbol{\nabla}(\mathbf{e}_z \cdot (\nabla \times \boldsymbol{\xi})) + \frac{R}{2}\mathbf{e}_z \times \boldsymbol{\nabla}\left[(\nabla \cdot \boldsymbol{\xi})^2 + \frac{\partial \xi_\alpha}{\partial x_\beta}\frac{\partial \xi_\beta}{\partial x_\alpha}\right], \qquad (1)$$

where $\boldsymbol{\xi} = \{\xi_x, \xi_y\}$ is the two-dimensional deformation (displacement) vector of the vortex crystalline medium treated as a continuous medium and summation is performed over repeated Greek indices. Equation (1) can be regarded as an analog of the acoustic equation for ordinary crystals. In [2], its linear properties were mainly analyzed. Below, we will study specific nonlinear properties of Eq. (1).

First, we will comment on this basic equation (see [2] for details). It is derived through a series expansion in the vortex displacement from the lattice points. Therefore, Eq. (1) is valid when the deformation is small ($|\boldsymbol{\xi}| \ll a$, where a is the distance between the neighboring lattice points; more precisely, it is the difference between the displacements of the neighbors that must be small) and, accordingly, the nonlinearity is small. By the point vortex, we mean a vortex whose core (a domain with a nonzero curl of the generalized electron momentum) is small compared to a. Microscopic vortices are characterized by their (identical) intensities q_0 and the screening-type flux function $\psi(|\mathbf{r}|/b)$ with the screening scale $b \gg a$. Both these parameters describe the flow produced by each vortex (for a vortex located at $\mathbf{r} = 0$, we have $\mathbf{v} = q_0\mathbf{e}_z \times \boldsymbol{\nabla}\psi$) and the vortex interaction energy $\mathscr{E} = q_0^2\sum_{i<j}\psi(\mathbf{r}_i - \mathbf{r}_j)$, where summation is performed over the entire lattice. In the continuous medium approximation adopted here, the sum is replaced with an integral over a plane (see [2]). The local nature of Eq. (1) (although, as follows from the form of the function ψ, the motion of each vortex is determined by its neighbors, which are $\sim(b/a)^2$ in number) is associated with the fact that the characteristic perturbation wavelength meets the inequality $\lambda \gg b$. The lattice elasticity moduli, which characterize the response of the lattice to uniform compression and torsion (shear), are time-independent (depend on its unperturbed structure alone),

$$R = \frac{2q_0}{\sqrt{3}a^2}\int \psi dxdy, \quad D \sim q_0|\psi(a)|$$

and differ significantly from each other: $D \sim R(a/b)^2$. The reason is that the discrete lattice demonstrates poor compressibility in vortex flows. For electron vortices in the electron magnetohydrodynamics [3], using the results of [2] and the pioneering work on vortex lattices [5], we obtain

$$R = \frac{4\pi q_0}{\sqrt{3}}\left(\frac{b}{a}\right)^2, \quad D = \frac{q_0}{8}$$

(the same behavior is typical of vortexes in superconductors [4], in which ψ is the Macdonald function $K_0(r\omega_{pe}/c)$).

It can be seen that, within Eq. (1), nonlinear waves can be analyzed in terms of the normal coordinates of shear and compression deformations, $\mathbf{e}_z \cdot (\nabla \times \boldsymbol{\xi})$ and $\nabla \cdot \boldsymbol{\xi}$.[1] For nonlinear effects, it turns out to be more convenient to consider potentials of these deformations by representing the displacement vector as $\boldsymbol{\xi} = \mathbf{e}_z \times \boldsymbol{\nabla}\phi + \boldsymbol{\nabla}\varphi$.

[1] The fact that only one type of waves that interrelate these deformations exists in two-dimensional media (unlike ordinary crystals, in which they evolve independently) is caused by Cartesian rather than Newtonian vortex mechanics (the position of a particle determines its velocity rather than acceleration). As a result, the two-dimensional vortex problem is, in a sense, equivalent to an ordinary one-dimensional problem (see [2]).

1063-780X/04/3003-0214$26.00 © 2004 MAIK "Nauka/Interperiodica"

Interestingly, the Schrödinger character of the phonon spectrum of a triangular electron vortex lattice ($\omega \propto k^2$) can be emphasized by combining these potentials into the wave function $\Phi = \phi + i\sqrt{R/D}\varphi$, which reduces the linear portion of Eq. (1) to:

$$i\frac{\partial\Phi}{\partial t} + \sqrt{RD}\Delta\Phi = 0$$

(removing the ∇ operator from both sides of the equation produces constant terms, which are hereafter set equal to zero; this, however, has no effect on the physically observed quantity $\boldsymbol{\xi}$). This expression also clearly demonstrates that, since in linear waves the ratio D/R is small, shear deformations dominate over uniform compression: $\phi \sim b/a\varphi$ (cf. [2]).

Let us start our nonlinear analysis with the study of stationary weakly nonlinear traveling acoustic waves. For these waves, an important effect is that the nonlinearity depends on the shape of the leading edge. In fact, it can easily be seen that the terms of Eq. (1) that are quadratic in $\boldsymbol{\xi}$ contain both ϕ and φ. However, for plane waves, $\boldsymbol{\xi}(x - ut)$, because of the geometrical degeneration, the much stronger (due to the deformation hierarchy $\phi \gg \varphi$ indicated above) nonlinearity ϕ^2 (as well as the weaker nonlinearity $\phi\varphi$) does not contribute to the equation

$$\varphi'' - (\varphi'')^2 + \frac{u^2}{RD}\varphi = 0 \quad (2)$$

(here, the prime stands for the derivative with respect to the independent argument described above), which follows from Eq. (1). As a result, this equation contains only the weakest nonlinearity related to φ^2. Single integration of Eq. (2) (taking into account the smallness of the nonlinear term) yields the following equation for cnoidal waves (i.e., waves that can be represented in terms of elliptical functions; there are no solitons here):

$$\varphi'^2 + \frac{u^2}{RD}\varphi^2 - \frac{2}{3}\left(\frac{u^2}{RD}\right)^2\varphi^3 = \text{const}. \quad (3)$$

The form of this equation is quite typical. However, for nonplanar wave fronts, nonlinear torsion effects are “switched on.” These effects, as can easily be seen, become dominant at $\lambda/r \gg (a/b)^2$, where r is the radius of curvature of the wave front [cf. Eq. (2) and (4)]. Strictly speaking, a curved traveling wave is unsteady; however, at $\lambda \ll r$, the unsteady behavior of the wave is not pronounced and the evolution of perturbations of the form $\boldsymbol{\xi}(r - ut)$ (where r is the radius in polar coordinates) can be studied using the equation

$$\phi'' + \frac{u^2}{RD}\left(\phi + \frac{R}{2ur}\phi'^2\right) = 0, \quad (4)$$

which follows from Eq. (1) when the above inequalities are satisfied. Here, the radius r in the coefficient by the nonlinear term can be considered constant. Unlike usual equation (2), new equation (4) is sensitive to the sign of u (i.e., whether the traveling wave is converging or diverging) and the sign of q_0 (i.e., the twist direction of microscopic flows in the crystal). Single integration now yields a more exotic relationship for weakly nonlinear cnoidal waves

$$\phi'^2 + \frac{u^2}{RD}\phi^2 - \frac{u^3}{3RD^2r}\phi^3 = \text{const}\left(\frac{u}{Dr}\phi - 1\right), \quad (5)$$

which cannot be reduced to an ordinary Sagdeev potential.

The strongly nonlinear evolution described by Eq. (1) is of particular interest. The possibility of this kind of evolution, in spite of the above assumptions used in deriving this equation, is also associated with the small value of a/b. In fact, the nonlinearity appears when quadratic (with respect to the deformation) corrections are taken into account in the first term on the right-hand side, which is proportional to R. This term can significantly exceed the second (caused by dispersion) linear term, which is proportional to D. For purely shear deformations ($\varphi \equiv 0$), the largest term vanishes, which yields the following equation ($D \longrightarrow 0$):

$$\frac{\partial\phi}{\partial t} = R\left[\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial y^2} - \left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2\right], \quad (6)$$

which contains the Hessian on its right-hand side and has a very symmetric form: Eq. (6) remains unchanged under the scaling transformations $x \longrightarrow \alpha x$, $y \longrightarrow \beta y$, $\phi \longrightarrow \gamma\phi$, and $t \longrightarrow \alpha^2\beta^2/\gamma t$ and under any rotation of the xy coordinate system. Although the approximation $D = 0$ is physically justified, it significantly changes the form of the equation, so that the evolution no longer has a wave character. For ordinary continuous media, this is equivalent of completely ignoring the crystal elasticity (although we partially take it into account here) or the temperature (i.e., pressure) of an ideal gas (see below).

Equation (6) can be written in terms of the variational derivative as

$$\frac{\partial\phi}{\partial t} = R\frac{\delta\mathcal{U}}{\delta\phi},$$

where

$$\mathcal{U} = \int\phi_x\phi_y\phi_{xy}dxdy,$$

or, in a more invariant form,

$$\mathcal{U} = -\frac{1}{4}\int(\nabla\phi)^2\Delta\phi dxdy.$$

It can easily be seen from these expressions that, since the density $\mathcal{U}$ is a homogeneous function with respect to derivatives in ϕ, solutions to Eq. (6) satisfy the following evolutionary relationships:

$$\frac{d}{dt}\int\phi^2dxdy = 6R\mathcal{U},$$

$$\text{or} \quad \frac{d^2}{dt^2}\int\phi^2dxdy = 6\int\left(\frac{\partial\phi}{\partial t}\right)^2dxdy.$$

Further, the right-hand side of Eq. (6) is a meaningful block of the Monge–Ampére equation [6], associated with the differential geometry of surfaces. This block vanishes as the Gaussian curvature of the surface $z = \phi(x, y)$ vanishes. In other words, a nonuniform deformation of the vortex crystal is static if ϕ describes a developable surface (a cylindrical or conical surface or a surface produced by tangents to an arbitrary three-dimensional curve). The general parametric expression for such ϕ has the form [6]

$$\phi = \zeta x + f(\zeta)y + g(\zeta), \quad x + f'(\zeta)y + g'(\zeta) = 0,$$

where $f(\zeta)$ and $g(\zeta)$ are arbitrary functions. In fact, since the physically observed quantity is the deformation $\boldsymbol{\xi}$ itself (rather than its potentials), the configurations for which the Hessian of ϕ is constant are also static. They can also be described by a general parametric expression, but this is only possible when this constant is negative [6].

If we expand the right-hand side of Eq. (6) in a power series in x and y, then the nontrivial evolution starts with the emergence of linear terms. Moreover, if $\phi_{xx}\phi_{yy} - (\phi_{xy})^2 = c_1x + c_2y$, then the deformations can easily be seen to increase by a linear ballistic law:

$$\boldsymbol{\xi}(x, y, t) = \boldsymbol{\xi}(x, y, 0) + (c_1\mathbf{e}_y - c_2\mathbf{e}_x)t. \tag{7}$$

There is a very wide range of initial deformation configurations (not only power functions) that produce such an evolution [6]. For example, at $c_1 = -1$ and $c_2 = 0$ (in view of the symmetries of Eq. (6), this does not limit the generality of our consideration), a suitable initial condition in the region $x > 0$ is

$$\phi_0 = \pm\frac{2}{3}x^{3/2}y + f(x) + C_y, \tag{8}$$

where the function f and constant C are arbitrary. Of course, such deformations grow without limit as $x \longrightarrow \infty$ (seemingly, such behavior is unavoidable in this case, because it is only singularities at infinity that can allow a nonuniformly deformed crystal to be displaced as a whole). However, because of the local character of the equations, if condition (8) holds only over a bounded region, then the evolution described by Eq. (7) can last here over a fairly long time.

As a rule, the greatest amount of information about the evolution described by nonlinear equations can be obtained from studying the possible singularities that exhibit themselves in a finite time. Three types of such singularities can be distinguished (since Eq. (6) is local, we expand ϕ about the singularity point (0, 0)):

$$\phi = -\frac{x^2y^2}{12R(t_0 - t)}, \quad \phi = \frac{(x^2 + y^2)^2}{48R(t_0 - t)}, \quad \phi = -\frac{(x - Ay^2)^3}{36RA(t_0 - t)}. \tag{9}$$

However, as $t \longrightarrow t_0$, higher order nonlinearities and/or dispersion effects prevent the deformations from growing without limit. The last singularity in (9) differs from the others in that it appears on a line rather than at a point. It is also of interest because, in the variables $\zeta = x - Ay^2$ and t, the one-dimensional Hessian is autonomous with respect to the independent argument and Eq. (6) transforms to the solvable equation

$$\frac{\partial\phi}{\partial t} = -2AR\frac{\partial\phi}{\partial\zeta}\frac{\partial^2\phi}{\partial\zeta^2},$$

which reduces to the classical nonlinear diffusion equation through the simple change of variables $\phi_\zeta = T$ (it is relevant to refer here to the excellent handbook [6]):

$$\frac{\partial T}{\partial t} = -2AR\frac{\partial}{\partial\zeta}\left(T\frac{\partial T}{\partial\zeta}\right). \tag{10}$$

Negative values of ART lead to the well-known monotonic spread of the T profile according to the attractive self-similarity law, i.e., to the disappearance of the perturbation, whereas in the case of $ART > 0$, the profile collapses in an explosive manner, which generalizes the last local singularity in (9):

$$T = \frac{1}{6[2AR(t_0 - t)]^{1/3}}\left(\zeta_0^2 - \frac{\zeta^2}{[2AR(t_0 - t)]^{2/3}}\right) \tag{11}$$

at $\zeta^2 < \zeta_0^2\,[2AR(t_0 - t)]^{2/3}$ and $T = 0$ outside the indicated interval. In fact, solution (11) describes both these cases (at $t > t_0$ and $t < t_0$, respectively). The amplitude of T at $\zeta = 0$ (i.e., the parameter ζ_0) is uniquely determined by the integral

$$\int_{-\infty}^{+\infty} T d\zeta = \frac{4}{3}\zeta_0^3,$$

which remains constant in the course of evolution. Peaking solution (11) describes the explosive growth of shear deformation along the lines ζ = const with the same torsion (clockwise or counterclockwise) as that of the flow produced by an individual vortex of the lattice.

Let us stress once again that the applicability of Eq. (6) to real lattices is limited in space (because of problems arising at $|x|, |y| \longrightarrow \infty$) and time (because we ignored lattice rigidity with respect to shear). The increase in the magnitude of the deformation vector $\boldsymbol{\xi}$ and the decrease in its scale length near the singularity points also violate the approximations of small nonlinearity and locality ($\lambda \gg b$). At the same time, the evolution described by Eq. (6) is non-Hamiltonian (which clearly exhibits itself in the case of diffusion described by Eq. (10)) with a tendency toward single-sided growth (which is seen from the fact that $d(RU)/dt > 0$), which raises the question of the problem of energy conservation in the system. However, it can be seen that, by the definition of this energy for Eq. (6) (taking the unperturbed state of the lattice as the reference state), it is equal to

$$\mathscr{E} = q_0 R \int \left[\frac{\partial^2 \phi}{\partial x^2} \frac{\partial^2 \phi}{\partial y^2} - \left(\frac{\partial^2 \phi}{\partial x \partial y} \right)^2 \right] dx dy,$$

because, since ψ simultaneously contributes to the dynamics and energy of the vortex ensembles, their energy density is proportional to the term under the operator $\mathbf{e}_z \times \nabla$ in the dynamic equation for $\boldsymbol{\xi} = \mathbf{e}_z \times \nabla\phi$. Since this energy density is proportional to the total derivative, the total energy of any localized perturbation is identically equal to zero (of course, only within this approximation). As for the “time arrow” in the direction of evolution, it is more correct to talk about a tendency toward the increase in $\nabla \times \boldsymbol{\xi}$ (i.e., $\Delta\phi$, which determines the sign of $\mathscr{U}$) in accordance with the microscopic characteristic q_0 (which determines the sign of R), because the preferred rotation direction of the continuous flow about certain vortexes destroys the chiral symmetry.[2]

[2] The change in the type of the equation when two Hamiltonian variables (ϕ and φ) are reduced to one is not unique. As a crude analogue of this effect in an ordinary medium, one can consider a one-dimensional (see above) flow of a compressible gas (for which these two variables are the potential of the velocity field and the density) at a zero temperature. The classic dynamic equation for the velocity $\partial v/\partial t + \partial(v^2/2)\partial x = 0$, which remains meaningful, can be rewritten for the potential $v = \varphi_x$ as $\partial\varphi/\partial t = -\varphi_x^2/2$, which quite clearly defines the time arrow for the functional $\tilde{\mathscr{U}} = -\int \varphi\varphi_x^2 dx$: $d\tilde{\mathscr{U}}/dt > 0$. Of course, the evolution described by Eq. (6) is much more complicated and diversified.

Thus, the above analysis has demonstrated that there is a qualitative difference between the nonlinear dispersion hierarchy of a vortex plasma (more precisely, the vortex ensemble in plasma) and that of a usual wave media described by equations like that of Korteweg–de Vries. Such peculiar behavior and its sensitivity to the internal chirality would seem to be of particular interest.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research (project no. 03-02-16765) and the Russian Federal Program for State Support of Leading Scientific Schools (project no. 2292.2003.2).

REFERENCES

1. B. N. Kuvshinov, J. Rem, I. J. Shep, and E. Westerhof, Phys. Plasmas **8**, 3232 (2001).
2. V. V. Smirnov and K. V. Chukbar, Zh. Éksp. Teor. Fiz. **120**, 145 (2001) [JETP **93**, 126 (2001)].
3. A. S. Kingsep, K. V. Chukbar, and V. V. Yan’kov, in *Reviews of Plasma Physics*, Ed. by B. B. Kadomtsev (Énergoizdat, Moscow, 1987; Consultants Bureau, New York, 1990), Vol. 16.
4. G. Blatter, M. V. Feigel’man, V. B. Geshkenbein, *et al.*, Rev. Mod. Phys. **66**, 1125 (1994).
5. V. K. Tkachenko, Zh. Éksp. Teor. Fiz. **50**, 1573 (1966) [Sov. Phys. JETP **23**, 1049 (1966)].
6. A. D. Polyanin and V. F. Zaĭtsev, *Handbook of Nonlinear Equations of Mathematical Physics* (Fizmatlit, Moscow, 2002), p. 247.

Translated by A.D. Khzmalyan