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Enhanced Superdiffusion and Finite Velocity of Levy Flights

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Abstract—A fractional differential equation is derived that describes the transformation of a stochastic transport from fast spreading ($\bar{x} \propto t^{\alpha}$, $\alpha > 1$) to a pseudowave regime ($\alpha = 1$) due to the finiteness of the velocities of individual particles. Qualitative features of the new regime are discussed. *© 2002 MAIK "Nauka/Interperiodica".*

1. INTRODUCTION

In this paper, we consider the diffusion, in a homogeneous and isotropic medium, of a macroscopic cloud of microscopic passive particles (i.e., those that do not affect the medium) characterized by a certain internal random-walk law. The latter circumstance relates this process to the class of stochastic transports, which is very popular in modern physics (see, for example, surveys [1–3] or the recent papers [4–9]). Depending on the features of the random walk at the microscopic level, macroscopic transport equations for the cloud density $n(x, t)$ may strongly differ from classical diffusion equations (while including the latter as a particular case) and involve, as a rule, fractional derivatives (see [10]) with respect to the space and/or time variables.

The standard random-walk model is as follows. Consider one-dimensional motion of a particle along a straight line x (multidimensional analogues will be considered at the end of this paper) that is characterized by the probability laws $g(|x|)$ and $f(t)$: the particles situated at any point (say, at x_0) may instantaneously jump to neighboring points, so that the probability that a particle occurs within the interval $(x_0 + x, x_0 + x + dx)$ is equal to $g(x)dx$; this jump occurs after a certain waiting period, so that the probability that a particle leaves its original position (the same point x_0) within the interval $(t, t + dt)$ (after arriving at this point) is equal to $f(t)dt$. It is the random character of the microscopic law of motion that is responsible for the stochastic character of the corresponding macroscopic transport: during this process, the initial state $n_0(x) = n(x, 0)$ is forgotten, and the distribution $n(x, t)$ attains a universal self-similar profile (see the cited literature and the reasoning below).

Historically, the first analyzed example of such walks was a crowd of drunk sailors with rather primitive g and f (see [1]); however, the model described here is fairly universal and admits a wide variety of physical interpretations. For example, we are especially interested in the resonance radiative transfer in a coronal plasma [11, 12]. In this case, the mean free path of a microscopic particle (a photon or a γ quantum) depends on whether this particle is emitted at the center or on the wing of a line, so that $g(x)$ is uniquely determined by the shape of the line contour, whereas $f(t)$ describes a spontaneous radiative decay of the excited state of the ion into which a quantum is transformed after its absorption. Of course, the actual process of radiative transfer is more complicated than the model under consideration (in particular, it requires that the plasma characteristics, such as the concentration and temperature, should be homogeneous and stationary); nevertheless, this model reflects many features of actual processes.

A specific problem to the solution of which this paper is devoted is the determination of the effect of the finiteness of a fixed velocity V of particles on the spreading of a cloud of excitations, i.e., taking into account the deviation from the standard model. However, a better understanding of the arising problems requires a brief account of the specific character of the process in the classical statement, i.e., with $V = \infty$. We will mainly follow [5], although there are many other works devoted to similar problems.

2. SPECIFIC FEATURES OF THE DESCRIPTION OF A STOCHASTIC TRANSPORT

The spreading of a cloud of particles is determined by the following important characteristics of the space and time distribution functions: the mean square of displacement (mean free path) and the mean expectation time

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 g(x)dx, \quad \langle t \rangle = \int_0^{\infty} t f(t)dt. \qquad (1)$$

When these parameters are finite, the effective transport equation asymptotically (i.e., for macroscopic time

$t \gg \langle t \rangle$ and spatial scales$|x| \gg \sqrt{\langle x^2 \rangle}$) reduces to the classical diffusion equation

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial x^2}, \quad D = \frac{\langle x^2 \rangle}{2\langle t \rangle}.$$

When these expressions are divergent (due to the slowly decaying power tails of g and f), the situation changes drastically. The case $\langle x^2 \rangle = \infty$ of the so-called Levy flights leads to the spatial nonlocality of the transport process (a fractional power of a Laplacian—a convolution-type integral operator with a certain power function of x—appears in the equation) and a faster spreading of the cloud, whereas the case with $\langle t \rangle = \infty$ (which is characterized by the term "traps") gives rise to the time nonlocality (a fractional time derivative of a certain different type!) and decelerates the macroscopic motion. In the general case, the asymptotic transport equation is expressed as

$$\frac{\partial^\gamma n}{\partial t^\gamma} = -K(-\Delta)^\beta n, \qquad (2)$$

where the constant K and the exponents $\gamma \le 1$ and $\beta \le 1$ are related to the powers of the tails of f and g, so that the evolution of the cloud width $n(x, t)$ is described by

$$\bar{x} \propto t^\alpha, \quad \alpha = \frac{\gamma}{2\beta}. \qquad (3)$$

The redundant minuses of the Laplacian are attributed to the form of the corresponding operator in the Fourier space (see below).

The standard terminology classifies any stochastic processes described by (3) with $\alpha > 1/2$ and $\alpha < 1/2$ as super- and subdiffusion processes.[1] Here, the stochasticity, or forgetting the initial conditions mentioned above, consists in ascribing the self-similarity of the Green's function,

$$G(x, t) = (1/t^\alpha)\Phi(x/t^\alpha),$$

of Eq. (2) to the general solution of this equation; namely, as soon as the characteristic width of G, which increases according to (3), becomes greater than the initial size of the cloud (one often speaks of the sufficiency of the doubling of the scale during the spreading process), the density distribution

$$n(x, t) = \int_{-\infty}^{+\infty} n_0(x')G(x - x', t)dx' \qquad (4)$$

becomes more and more universal:

$$n(x, t)|_{t \to \infty} \longrightarrow G(x, t)\int_{-\infty}^{+\infty} n_0 x' dx'$$

(see below). In other words, the self-similarity, which generally facilitates the analysis of the properties of mathematical physics equations, in this case is "attracting" in addition; this considerably simplifies the analysis of the possible behavior of such physical systems. Interestingly, the specific form of $G(\Phi)$ depends on β and γ separately, rather than on α.

Thus, it is the Levy flights that are responsible for the superdiffusion behavior in this model.[2] Depending on a specific physical problem (for example, depending on the shape of the line contour), the parameter α may take any value. Here, the case $\alpha > 1$ is of greatest interest (when $\langle t \rangle \neq \infty$, the boundary of α is associated with the divergence of the moment $\langle |x| \rangle$ of g) when the spreading of the cloud occurs at increasing rate. There is no established term for this situation; the term "enhanced superdiffusion" used in the title sounds fairly natural.

As we mentioned above, the problems presented in this section were analyzed from various viewpoints and to different degrees of comprehensiveness in many studies. Nevertheless, the question concerning such an important generalization of the random-walk model as the consideration of the finiteness of the velocities of microscopic particles as they move to neighboring points (which is certainly inherent in real physical situations) has scarcely been analyzed. As applied to nondiffusion equations, this question has been raised quite recently in [6, 7]; however, one can hardly agree with all the assertions made in those works.

3. THE EFFECT OF THE FINITENESS OF FLIGHT VELOCITIES: PRIMARY CONSIDERATIONS

In fact, even a qualitative analysis of the arising new situation (cf. [7]) allows one to make important assumptions about the expected phenomena. First of all, we have to realize what we are going to find out. The obvious effect of the finiteness of the flight velocity is the fact that the Green's function of the effective equation (which represents the density distribution $n(x, t)$ in (2) with $n_0 = \delta(x)$; see (4)) identically vanishes for $|x| > vt$. This fact may be very important, for example, in physical problems that require a rigorous consider-

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[1] In fact, here we implicitly assume that physical systems are spatially and temporally homogeneous, as was mentioned in the Introduction, since a conventional diffusion process with $\bar{x} \sim \sqrt{Dt}$ in the case of $D = D(x, t)$ can guarantee the fulfillment of (3) for any α (which frequently occurs).

[2] In general, the converse is not true: the fact that a transport equation contains the operator $(-\Delta)^\beta$ does not always guarantee that there are Levy flights in the physical phenomenon described by this equation. For example, in the problems of skin effect [13] or the Maxwell relaxation of charge [14] in thin films, the operator $(-\Delta)^{1/2}$ is attributed to purely geometric reasons, and there are no microscopic particles at all; in these problems, n plays the role of the component of a magnetic (electric) field normal to the film.

ation of the relativistic causality principle, and does not always strongly influence the behavior of the main group of particles. When $\alpha < 1$, the boundary $G \equiv 0$ moves faster than the characteristic self-similar parameter (3), so that the old Green's function is formed asymptotically (Eq. (2) also makes sense only asymptotically) with the distortion of $\Phi(\xi)$ only on "far" tails (the boundary of these distortions being such that $\xi_{\text{bound}}|_{t\to\infty} \longrightarrow \infty$). We will not consider this phenomenon. A sufficiently detailed discussion of this phenomenon as applied to the conventional diffusion equation (the Gaussian profile $\Phi(\xi)$) was given, for example, in [15].

The situation is entirely different for the enhanced superdiffusion. Here, in contrast, the evolution of the self-similar width of the cloud is faster; therefore, the asymptotic condition $|x| < vt$ substantially changes the form of G and the structure of Eq. (2) itself. The present paper is devoted precisely to the derivation of this equation with allowance for these new circumstances.

It is fair to say that, when $\alpha < 1$, the finiteness of v may change the value of the coefficient K, while leaving unchanged (in the above sense) the fractional exponents of (2). It seems sufficiently obvious that, when $\langle t\rangle$ and $\langle|x|\rangle$ are finite, the following change is made in K:

$$\langle t\rangle \longrightarrow \langle t\rangle + \frac{\langle|x|\rangle}{v}. \tag{5}$$

This fact is well known for a diffusion process. However, the change (5) should also be valid for infinite values of the mean expectation time (in the sense that finite values of $\langle|x|\rangle$ do not influence the value of K any longer).

The earlier works [6, 7] shed light differently on the v phenomena discussed. There is a rather enigmatic assertion in [6] that the finiteness of v reduces any equations of type (2) (with any β and γ) to a diffusion equation. This error was corrected in [7], which was only partly concerned with this problem; however, the authors of [7] restricted themselves to the derivation and analysis of the type (2) equations with regard to (5). As for the enhanced superdiffusion (recall that $\langle|x|\rangle = \infty$ in this case), a strange result was obtained in [7] that, since the finiteness of v for $\alpha > 1$ qualitatively changes the Green's function of the initial equation (2), this equation is "absolutely inapplicable to the description of real processes"; as a result, the strongest effect due to the finite particle velocity ($v \neq \infty$) was left unstudied.

In fact, the more a power function with a large exponent leads a linear function for large values of the argument, the more it falls behind it for small values of the argument. Under enhanced superdiffusion, a cloud first spreads rather slowly, so that the constraint $|x| < vt$ starts to influence the self-similarity (3) not very soon and, for sufficiently large values of v, the transformation of the process, which occurs for $t \propto v^{1/(\alpha-1)}$, may leave enough time for the initial macroscopic evolution by the law (2), (3) (intermediate asymptotics).

A clear physical example of such a possibility is provided by a radiative transfer in a coronal plasma. This phenomenon is characterized by considerable values of α and huge values of v (it is the velocity of light!). The latter fact allows one to set v equal to infinity in the majority of plasma problems; however, technically, this is not always correct. Moreover, this simply may prove to be incorrect even in the present statement, let alone in other possible physical realizations of the model; this fact stimulates the analysis of the transformation described. The mathematical procedures involved will be based on the approach used in [5].

4. INITIAL EQUATIONS

An adequate description of the kinetics of a transport process requires the introduction of several new parameters N, F, and Q in addition to n, g, and f. As we noted in the Introduction, the particles located at a given point x remember the moment when they arrived at this point; therefore, their spatial density n represents an integral of a certain distribution N with respect to lifetime τ:

$$n(x, t) = \int_0^\infty N(x, t, \tau)d\tau.$$

It is more convenient to express the transition to further motion in terms of the probability to survive until τ,

$$F(\tau) = 1 - \int_0^\tau f(t)dt,$$

rather than directly in terms of f. Finally, we can denote by $Q(x, t)$ a flow emanating from a given point and directed to either side and reaching any distance. According to the definition, f characterizes the escape rate in terms of particles that have initially arrived at this point, of which only a part determined by $F(\tau)$ remain by the moment τ. Therefore, by the conditional probability formula (see [5]), we have

$$Q(x, t) = \int_0^\infty \frac{N(x, t, \tau)}{F(\tau)} f(\tau)d\tau. \tag{6}$$

Thus, we obtain the following compact expression for the equation of balance for the particles that are located at a given point at a given moment (for resting particles):

$$n(x, t) = \int_{-\infty}^{+\infty} g(x')\theta\left(t - \frac{|x'|}{v}\right)$$

$$\times \int_0^{t - |x'|/v} Q\left(x - x', t - \frac{|x'|}{v} - t'\right)F(t')dt'dx' \quad (7)$$

$$+ \int_t^{\infty} \frac{N_0(x, \tau - t)}{F(\tau - t)}F(\tau)d\tau,$$

where $N_0(x, \tau) \equiv N(x, 0, \tau)$ is the initial lifetime distribution of particles. Equations (6) and (7) provide a full description of the situation.

It is easily seen that, while the parameters F and Q are only needed to make the expression more compact, the introduction of the distribution N is essential in the sense that the equation for n cannot be expressed only in terms of the density n of particles; i.e., in general, the macroscopic kinetics depends on the microscopic details in a rather awkward way. Unfortunately, this technical feature is not properly reflected in the literature; as a rule, one immediately writes only a microscopic transport equation (see, for example, [7]), which is actually valid only in a certain asymptotic sense.

One meets no problems only in the case

$$f = \mu\exp(-\mu t), \quad \mu = 1/\langle t\rangle,$$

when F and f (and, consequently, Q and n) are just proportional to each other. This law is encountered quite frequently in physical applications; for example, it is typical of the radiative decay of excited states. For other f (which are also encountered in practice), the features of the function of N versus its arguments make their own contributions. The point is that the newly arriving particles form a self-similar profile

$$N_{\text{new}} = \theta(t - \tau)P(t - \tau)F(\tau)$$

with a correlated dependence on t and τ, where P is an incoming flow. The old initial distribution is shifted to the domain $\tau > t$ and monotonically decreases due to the flights to the neighboring points (see (7)). When such self-similarity occupies a greater part of the profile $N(\tau)$ and starts to dominate in Q, the integrals in dt' and $d\tau$ (the latter integral enters the definition of Q (6)) in (7) can be interchanged, and, instead of N, one obtains its integral—the macroscopic density n. For simplicity, one can choose N_0 as a shifted Dirac's delta function (as proposed in [5]):

$$N_0 = n_0\delta_+(\tau), \quad \int_0^{\infty} \delta_+(\tau)d\tau = 1.$$

This provides a self-similar relation between the functions of N versus t and τ from the very beginning of the process. Then, instead of (6) and (7), one obtains a single basic equation (cf. [5] with $v = \infty$ and [7]):

$$n(x, t) = \int_{-\infty}^{+\infty} g(x')\theta\left(t - \frac{|x'|}{v}\right)$$
$$\times \int_0^{t - |x'|/v} f(t')n\left(x - x', t - \frac{|x'|}{v} - t'\right)dt'dx' + F(t)n_0(x). \quad (8)$$

In the general case, this equation is valid only asymptotically; this makes the problem of describing the evolution of a cloud for small time $t \longrightarrow 0$ rather involved but does not influence the present study. In fact, Eq. (8) in the problem considered is not sufficiently asymptotic. For our purposes, we have to make the transition $t \longrightarrow \infty$ (which is accompanied by the transition $\bar{x} \longrightarrow \infty$ due to the spreading phenomenon). The simplest way to do this is to apply the Laplace transform with respect to the time variable and the Fourier transform with respect to the space variable; this will save us from dealing with convolution integrals; calculations with functions are simpler than those with operators.

5. ASYMPTOTIC EVOLUTION

To facilitate intermediate calculations and to make definite certain numerical coefficients, it is desirable to specify the expressions for the distribution functions g and $f(F)$ without losing the possibility to describe a variety of forms of Eq. (2). The following class of functions proves to be very convenient (see [5]; to simplify the cumbersome expressions, everywhere below we use dimensionless variables, so that x, $t \sim 1$ correspond to microscopic scales):

$$g(x) = \frac{\Gamma(\beta + 1/2)}{\sqrt{\pi}\Gamma(\beta)} \frac{1}{(1 + x^2)^{\beta + 1/2}},$$
$$f(t) = \frac{\gamma}{(1 + \gamma)^{\gamma + 1}}, \quad F(t) = \frac{1}{(1 + t)^{\gamma}}. \quad (9)$$

Here, Γ is the Euler gamma function, and the numerical coefficients are determined by the normalization of g and f to 1. Only the power exponents of the tails of functions that are parameterized by the positive numbers β and γ are essential for the further analysis. These exponents coincide with those introduced in Eqs. (2) and (3) (variants (9) with β, $\gamma > 1$ give standard integral-order derivatives in macroscopic transport equations[3]).

[3] Stochastic transports of type (2) and with a fractional derivative with $\gamma > 1$ are yet possible but in entirely different problems, for example, in a drift of a passive impurity by a turbulent flow of a fluid [16].

As reference information, we present the aforementioned power moments of these distributions (as long as they exist):

$$\langle x^2 \rangle = \frac{1}{2(\beta-1)}, \quad \langle |x| \rangle = \frac{\Gamma(\beta-1/2)}{\sqrt{\pi}\Gamma(\beta)},$$

$$\langle t \rangle = \frac{1}{\gamma-1}.$$

Below, we will restrict the analysis to the case $\beta < 1$ (the opposite case is described by analogous formulas) and mainly to $\beta < 1/2$.

The simultaneous application of the above Laplace and Fourier transforms reduces Eq. (8) to

$$n_{pk} = [g(x)\exp(-p|x|/v)]_k f_p n_{pk} + F_p n_{0k} \qquad (10)$$

(here, the symbol $[\cdot]_k$ stands for a Fourier component of an appropriate function). The asymptotic transition $|x|$, $t \longrightarrow \infty$ in (8) corresponds to the dual transition k, $p \longrightarrow 0$ in (10). For the latter transition, it suffices to expand the equation in power series in k and p and retain the first terms in the expansion. Here, it is convenient to subtract the expression $f_p n_{pk}$ from the left- and right-hand sides of (10) simultaneously. Then, for g and f from (9), the original Eq. (10) is transformed to (the appropriate integrals and their series expansions can be found, for example, in [17])

$$\begin{aligned}\left[p^{\gamma}\Gamma(1-\gamma) + \frac{p}{\gamma-1}\right]n_{pk} = -\Bigg\{\frac{\Gamma(\beta-1/2)}{\sqrt{\pi}\Gamma(\beta)}\frac{p}{v} \\ + \frac{\Gamma(1-\beta)}{\Gamma(1+\beta)}\frac{1}{2^{2\beta+1}\cos(\pi\beta)} \\ \times\left[\left(\frac{p}{v}+ik\right)^{2\beta} + \left(\frac{p}{v}-ik\right)^{2\beta}\right]\Bigg\}n_{pk} \\ + \left[p^{\gamma-1}\Gamma(1-\gamma) + \frac{1}{\gamma-1}\right]n_{0k}\end{aligned} \qquad (11)$$

(cf. [5] and (2) with $v = \infty$).

Actually, for every specific value of β and γ, Eq. (11) has a simpler form. Say, for $\gamma > 1$ (i.e., for finite $\langle t \rangle$), one can neglect $p^{\gamma-1}$ as compared with unity (p^0) in square brackets on both sides of the equation, whereas, for $\gamma < 1$, the situation is opposite. The cases $\beta > 1/2$ and $\beta < 1/2$ give similar results. The "critical" values of the numbers $\beta = 1/2$ and $\gamma = 1$ (as well as $\beta = 1$) yield a slightly more complicated problem (which requires removal of the $\infty - \infty$ uncertainties), since there appear logarithms in addition to the power terms in the equation (see [5]). The terminology of $\partial^{\gamma}/\partial t^{\gamma}$ and $(-\Delta)^{\beta}$ used in (2) is attributed to the coefficients p^{γ} and $|k|^{2\beta}$ (for $v = \infty$).

The combination containing $(p/v \pm ik)^{2\beta}$ (just as the first term on the right-hand side, where the terms $\pm ik$ are merely canceled out) results from the series expansion of the integral

$$\int_{-\infty}^{+\infty} \frac{\exp(-p|x|/v)\cos(kx)}{(1+x^2)^{\beta+1/2}}dx. \qquad (12)$$

Therefore, we have to single out a branch with real values for real k and positive p; i.e., the above combination can be represented as

$$2\left|\frac{p^2}{v^2}+k^2\right|^{\beta}\cos(2\beta\varphi), \quad \cos\varphi = \frac{p/v}{\sqrt{(p/v)^2+k^2}}. \qquad (13)$$

It is clear that, for finite values of $\langle |x| \rangle$, the situation actually coincides with the case discussed in Section 3. For example, if $\langle t \rangle$ is also finite, then the first term in curly brackets on the right-hand side of (11) is combined with the expression on the left-hand side to give a term of the form

$$(\langle t \rangle + \langle |x| \rangle/v)pn_{pk}$$

(cf. [7]). It is interesting to note that there is no such a coefficient on the right for n_{0k}; this naturally implies that the number of particles described by Eq. (11) asymptotically decreases by a factor of

$$\langle t \rangle/(\langle t \rangle + \langle |x| \rangle/v)$$

as compared with its initial value. This situation has a simple explanation: The finite velocity of motion results in a natural separation of particles into two types: resting ones, which are described by Eqs. (8), (10), and (11), and flying ones. It is obvious that, on macroscopic temporal and spatial scales satisfying the inequalities

$$t \gg \langle t \rangle, \quad |x| \gg \langle |x| \rangle,$$

the densities of each type of particles are proportional to the time during which they stay in this state; therefore (cf. [7]),

$$n_{\rm fly}(x,t) = \frac{\langle |x| \rangle}{v\langle t \rangle}n(x,t). \qquad (14)$$

If there are flying particles in the system at the initial moment, then the following change is made in (11):

$$n_{0k} \longrightarrow n_{0k} + n_{{\rm fly}0k}.$$

This trivial circumstance results in another (in addition to those discussed in Section 3) nontrivial phenomenon when $\langle |x| \rangle = \infty$. In this case, the number of flying particles should asymptotically increase; in other words, an irreversible transformation of resting particles (whose number asymptotically tends to zero) into flying ones occurs during such a transport. Recall that these two states are actually quite different in a physical problem of radiative transfer through a plasma, so that the total number of excitations in a medium will decrease, while the number of γ quanta will asymptotically tend to a constant value. Naturally, this fact manifests itself in the structure of the Green's function since

$$\int_{-\infty}^{+\infty} n dx$$

is not an integral of motion any longer in the effective transport equation.

Thus, the problem in question is solved: Eq. (11) represents an asymptotic equation of stochastic transport with the finite velocity of Levy flights. In the next section, we discuss the characteristic features of the equation obtained.

6. TRANSFORMATION OF REGIMES UNDER ENHANCED SUPERDIFFUSION

In addition to the renormalization of the coefficient K in (2), which was pointed out in [7], for finite values of v, the fractional power of the Laplacian is replaced, according to (11), by a slightly more exotic combination

$$|k|^{2\beta} \longrightarrow \frac{(p/v + ik)^{2\beta} + (p - ik)^{2\beta}}{2\cos(\pi\beta)}. \quad (15)$$

In principle, within the framework of the terminology used, this combination can be expressed in terms of an appropriate sum of fractional derivatives

$$\left(\frac{1}{v}\frac{\partial}{\partial t} \pm \frac{\partial}{\partial x}\right)^{2\beta}$$

(which, possibly, more clearly demonstrates the necessary condition $G \equiv 0$ for $|x| > vt$). However, in contrast to $(-\Delta)^\beta$ or $\partial^\gamma/\partial t^\gamma$, this expression is not widely used. How does such a transformation of operators influence the structure of the Green's function of the transport equation?

Since we are interested in the change of the self-similarity (i.e., in a maximally strong influence; see Section 3), we have to study a self-similar relation between the characteristic scale of the wave vector $\bar{k}$ and p in Eq. (11). Let us rewrite (11) in a more compact form setting, for definiteness, $\gamma > 1$ and $\beta < 1/2$ (cf. (2)),

$$pn_{pk} = -K\frac{\cos(2\beta\varphi)}{\cos(\pi\beta)}\left(k^2 + \frac{p^2}{v^2}\right)^\beta n_{pk} + n_{0k},$$
$$K = \frac{\Gamma(1-\beta)}{\Gamma(1+\beta)}\frac{\gamma - 1}{2^{2\beta}}, \quad (16)$$

and assume that $v \gg 1$ for transformation of regimes at the stage of macroscopic evolution; formula (16) makes sense only for this stage since we assumed that $\bar{x}$, $t \gg 1$ when deriving these formulas.

Consider two limit regimes in (16): the old, now intermediate, asymptotics and the new, final, asymptotics. Initially (i.e., for moderately large t and moderately small p), the term $(p/v)^2$ can be neglected as compared with k^2 in the parentheses on the right-hand side of (16) (by setting $\varphi = \pi/2$). Indeed, in this case, (16) reduces to (2) with $\bar{x} \propto t^{1/2\beta}$ (see (3)) or $\bar{k} \propto p^{1/2\beta}$ (see the right- and left-hand sides of (16)), which allows us to neglect $(p/v)^2$ as compared with $p^{1/\beta}$ up to $p \sim v^{-2\beta/(1-2\beta)}$ (or always when $\beta > 1/2$) or $t \sim v^{2\beta/(1-2\beta)}$ ($t \sim v^{1/(\alpha-1)}$ when $\gamma < 1$; cf. Section 3).

However, after a certain period of time, the situation is radically changed, and the new self-similarity $\bar{x} \sim vt$ ($\bar{k} \sim p/v$) is established, so that the entire left-hand side of (16) can be neglected:

$$K\frac{\cos(2\beta\varphi)}{\cos(\pi\beta)}\left(k^2 + \frac{p^2}{v^2}\right)^\beta n_{pk} = n_{0k}.$$

Hence, the required Green's function of the "transformed" equation is expressed as

$$G_{pk} = \frac{\cos(\pi\beta)}{K\cos(2\beta\varphi)(k^2 + p^2/v^2)^\beta}. \quad (17)$$

This function really satisfies the announced conditions that it should identically vanish at large distances and that the total number of resting particles should asymptotically decrease. Since

$$\int_{-\infty}^{+\infty} n dx = n_k|_{k=0},$$

the effect is determined by the zero harmonic of G. In addition,

$$G_{p0} \propto p^{-2\beta},$$

therefore,

$$\int_{-\infty}^{+\infty} n dx \propto t^{2\beta - 1} \quad (\text{or } t^{2\beta - \gamma} \text{ at } \gamma < 1)$$

(recall that this regime is not realized when $\gamma < 2\beta$ or $\alpha < 1$!). This expression coincides with the expression obtained by integrating (14) under the following interpretation:

$$\int_{-\infty}^{+\infty} n_{\mathrm{fly}} dx \longrightarrow \mathrm{const}, \quad \langle|x|\rangle \longrightarrow \int_{-vt}^{vt} g(x)|x| dx \propto t^{1-2\beta},$$

$$\langle t\rangle \longrightarrow \int_0^t f(t) t dt \propto t^{1-\gamma}.$$

Note that, when passing from p to t, the exponents in the expressions for the real function G and the self-similar parameter $\bar{x}$ (cf. conventional diffusion equation) differ by unity.

The transformation of regimes from the conservation of the total number of particles to its decrease occurs, according to the general Eq. (16), by the following law:

$$\int_{-\infty}^{+\infty} n_p dx \propto \frac{1}{p + K(p/v)^{2\beta}}. \quad (18)$$

As is clear from (11), when $\gamma < 1$, the second term in the denominator of this expression is additionally multiplied by $p^{1-\gamma}$.

7. PARTICULAR CASES AND THE DISCUSSION OF GENERAL PROPERTIES

According to (4), asymptotic relation (17) (or its full variant following from (11) or (16)) allows one to obtain a solution to the problem of enhanced superdiffusion with the finite velocity of Levy flights for any initial distribution $n_0(x)$. However, operations in the Laplace–Fourier space prove to be poorly descriptive; therefore, to give an idea of the character of the new form of the Green's function, we present simple expressions for some of these functions in conventional variables.

For the case $\beta = 1/4$ ($\bar{x} \propto t^2$), which corresponds to the frequently encountered Lorentz contour of lines in the physical problem of radiative transfer, we have

$$G \sim \frac{\theta(vt - |x|)}{v^{1/2}t^{3/2}} \quad (19)$$

(recall that this expression is valid only for $t \gg v$). Here, Eq. (17) is inverted completely due to the simple relation between the cosines of a simple and a half angle. In this case, the transformation of regimes (18) occurs by the following law:

$$\int_{-\infty}^{+\infty} n dx \propto \exp\left(K^2 \frac{t}{v}\right) \operatorname{erfc}\left[K\left(\frac{t}{v}\right)^{1/2}\right].$$

Another case corresponds to a whole class of extremely fast transports with $\beta \longrightarrow 0$; here, one can set $\cos(2\beta\varphi) \longrightarrow 1$ (which is not fully rigorous since the questions concerning the convergence rate in dual spaces are rather complicated; nevertheless, $\varphi \sim 1$ in the self-similarity domain $\bar{k} \sim p/v$), and, hence,

$$G \sim \frac{v\theta(vt - |x|)}{(v^2t^2 - x^2)^{1-\beta}} \quad (20)$$

(for $t \gg v^{2\beta} \gg 1$, this inequality imposes certain constraints on β for a given v).

Despite the fact that the Green's functions obtained are similar to their analogues encountered in wave problems, the process considered possesses all the characteristic features inherent in stochastic transport. Indeed, we still deal with the infinitely spreading function

$$G = \frac{1}{t^{2-2\beta}} \Phi(x/t);$$

therefore, according to (4), any initial profile $n_0(x)$ tends to the following universal distribution involving few parameters (see [5]):

$$n(x, t) \approx AG(x - x_0)[1 + O(1/t^2)],$$

$$A = \int_{-\infty}^{+\infty} n_0 dx, \quad Ax_0 = \int_{-\infty}^{+\infty} x n_0 dx;$$

i.e., the forgetting fully manifests itself (cf. the Hamiltonian problem of a truly wave motion). Note that, since $n(x, t)$ represents a convolution of n_0 with G, the sharp gradients (and even discontinuities) of the latter for $|x| \sim vt$ in the asymptotic profile of the macroscopic concentration are actually smoothed over a distance on the order of the initial width of a cloud.

Nevertheless, variant (17) of the Green's function is different from those obtained in earlier investigations. When $\langle |x| \rangle \neq \infty$, for any velocity v, one can determine the boundaries starting from which the effective transport equation loses its sensitivity to the initial conditions; this occurs for $\bar{x} \gg \langle |x| \rangle$. In fact, we always deal with the transient regime. This fact manifests itself in the following. As was noted above, for $\beta > 1/2$ and $n_{\mathrm{fly}0}(x) \neq 0$, we obtain only a certain renormalization of the initial condition; however, in our case, this function does not fully describe the situation; an adequate description requires the introduction of the distribution n_{fly} at the place where the particles will settle (cf. the introduction of the distribution $N(\tau)$). Therefore, the whole subsequent evolution can be radically changed by an appropriate choice of the function $n_{\mathrm{fly}}(x)$.[4] In this case, the relation between the functions n and n_{fly} is different from a simple proportion even for $n_{\mathrm{fly}0} \equiv 0$ (see [7]):

$$n_{fly} = \frac{1}{2v} \int_{-\infty}^{+\infty} Q\left(x - x', t - \frac{|x'|}{v}\right) \times \theta\left(t - \frac{|x'|}{v}\right) \int_{|x'|}^{\infty} g(y) dy dx'.$$

This fact is attributed to the extreme nonlocality of the problem and the asymptotic extinction of resting particles.

[4] Note that, in the case of highly nonequilibrium initial distribution functions $N_0(\tau)$ for $\langle t \rangle = \infty$, the formation of the self-similar "$t - \tau$" profile of N may also take a rather long time.

8. INCREASING THE DIMENSION OF THE PROBLEM

It is well known that, in conventional superdiffusion (with $v = \infty$), the transformation of the effective equation when passing from a one-dimensional problem to a multidimensional problem is trivial: the fractional power of the Laplacian in Eq. (2) corresponds to $-|\mathbf{k}|^{2\beta}$ in space of arbitrary dimension (see [1–7]). The situation is completely different in the present case; it is rather difficult to rewrite (11) or (16) in a new form. Here, the problem is associated with the fact that the simple additivity of p and $\pm ik$ is attributed exclusively to the convenient properties of expression (12), which, in the general case, are not preserved in spaces of other dimensions.

Nevertheless, the situation is quite similar in the most frequently encountered three-dimensional case:

$$\int_{-\infty}^{+\infty} \exp(-pr/v)\exp(-i\mathbf{k}\cdot\mathbf{r})g(r)d\mathbf{r}$$

$$= -4\pi\int_{0}^{\infty}\frac{\exp(-pr/v)\sin(kr)}{k}g(r)rdr$$

(in the two-dimensional case, the sine is replaced by a Bessel function). Hence, for $v \neq \infty$, the block of terms containing $k^{2\beta}$ (where k is the modulus of the wave vector) in the effective equation is replaced by

$$\frac{(p/v + ik)^{2\beta+1} - (p/v - ik)^{2\beta+1}}{2ik\cos(\pi\beta)} \tag{21}$$

and the separation of the required branch yields $\sin[(2\beta + 1)\varphi]$ rather than a cosine. In terms of conventional physical variables, the situation is not so simple as in (15), since, although p/v still corresponds to $\partial/\partial t$, $\pm ik$ is an integral operator of the type $\Delta^{1/2}$ from the very beginning. However, from the mathematical viewpoint, operations with (21) are not substantially more difficult than operations with (15).

9. CONCLUSION

Thus, the solution of the problem on the determination of asymptotic properties of the stochastic transport of microscopic particles has allowed us to derive new macroscopic equations describing the kinetics of this process with allowance for the finiteness of the velocity of particles. Despite a different type of fractional derivatives involved, these equations prove to be very convenient for a sufficiently detailed analysis of the phenomena associated with the finiteness of v: the transformation of the self-similarity of the Green's functions, the extinction of resting particles, and nontrivial dependence of the system evolution on the dimension of the problem. All these questions can find direct practical application, in particular, to the study of radiative transfer in plasma.

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