- Kapitza pendulum using FM expansion:

$$H(t) = \frac{p_\theta^2}{2m} - m\left(\omega_0^2 + A\omega\cos\omega t\right)\cos\theta$$

$$= H_{kin} + H_{pot} + \omega f(t) H_{drive}$$

- where $H_{kin} = p_\theta^2/2m$

$$H_{pot} = -m\omega_0^2\cos\theta$$

$$H_{drive} = -m\cos\theta \quad , \quad f(t) = A\cos\omega t$$

→ plug in expressions: for $\tau := \omega t$

$$H^{(0)} = \frac{1}{2\pi}\int_0^{2\pi} d\tau\, H(\tau) = H_{kin} + H_{pot}$$

$$H^{(1)} \equiv 0 \quad , \text{ since } f(\tau) = f(2\pi - \tau)$$

$$H^{(2)} = \frac{\omega}{12\pi\omega^2}\left[H_{kin}, \left[H_{kin}, H_{drive}\right]\right] \times$$

$$\times \int_0^{2\pi} d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 \; 2f(\tau_2) - f(\tau_1) - f(\tau_3)$$

$\propto O(1/\omega) \hookrightarrow$ suppressed, drop

$$- \frac{\cancel{\omega^2}}{12\pi\cancel{\omega^2}}\left[H_{drive}, \left[H_{drive}, H_{kin}\right]\right]$$

$$\times \int_0^{2\pi} d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 \; f(\tau_2)f(\tau_3) + f(\tau_2)f(\tau_1) - 2f(\tau_1)f(\tau_3)$$

$\propto O(1)$

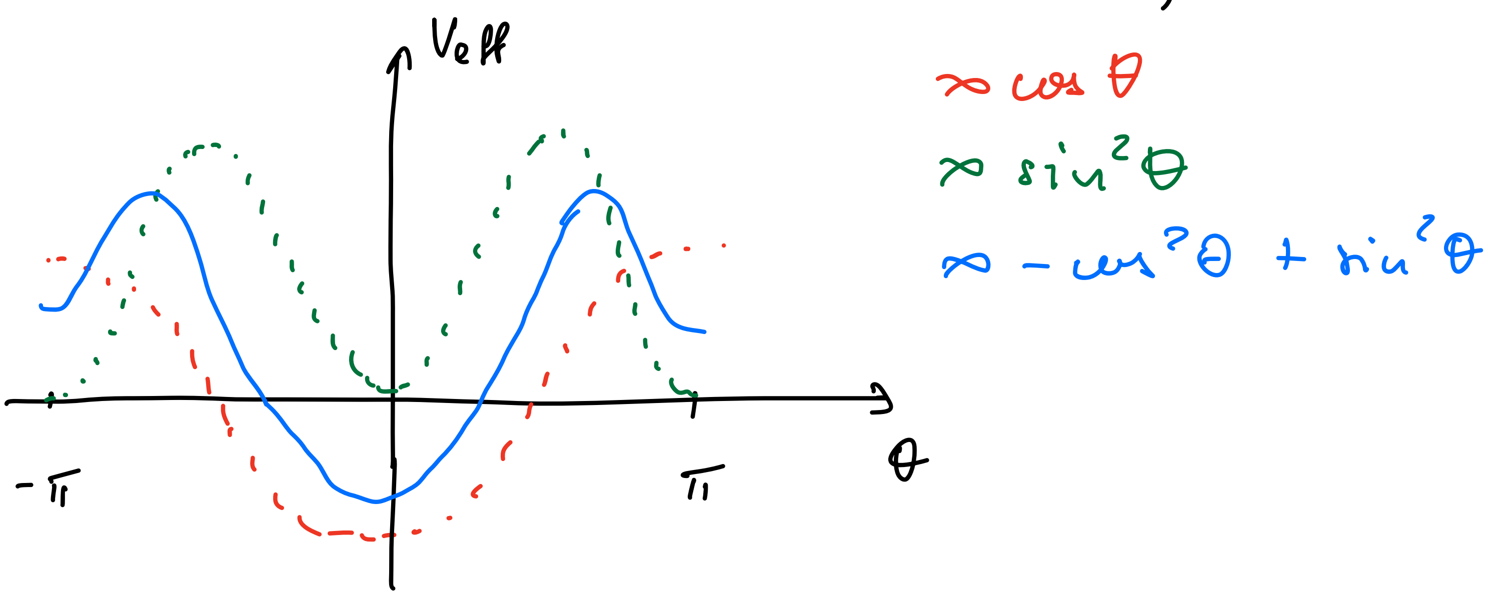
- can show: using $[p_\theta, f(\theta)] = -i\partial_\theta f(\theta)$

1) $[H_{drive}, [H_{drive}, H_{kin}]] = \frac{1}{m}\left(\frac{\partial H_{drive}}{\partial\theta}\right)^2$

2)
$$\frac{1}{(2\pi)}\int_0^{2\pi} d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 \; f(\tau_2)f(\tau_3) + f(\tau_2)f(\tau_1) - 2f(\tau_1)f(\tau_3) = \frac{A^2}{4}$$

$$\Rightarrow H_F^{(0)} = H_{kin} + H_{pot} + \frac{A^2 m}{4}\underbrace{(\partial_\theta H_{drive})^2}_{=\sin^2\theta}$$

$$= \frac{p_\theta^2}{2m} \underbrace{- m\omega_0^2\cos\theta + \frac{A^2}{4} m \sin^2\theta}_{=: V_{eff}(\theta) \text{ effective potential}}$$



$$\partial_\theta^2 V_{eff}(\theta) = m\omega_0^2\cos\theta + \frac{A^2}{2} m \cos 2\theta$$

$$\overset{\theta=\pi}{=} -m\omega_0^2 + \frac{A^2}{2} m \overset{!}{>} 0$$ stable for $A > \sqrt{2}\,\omega_0$ at inverted equilibrium

dynamical stabilization

- due to scaling of drive amplitude with frequency, need to be careful with power-counting!

→ limit $\omega \to \infty$ not just given by time-average

• are we missing any higher order terms?

a) n'th order term contains n-nested commutators

b) & n-fold t-order integral $\propto 1/\omega^{n-1}$

to have a contrib. of $\mathcal{O}(1)$, only nested commutators containing H_{kin} precisely once (& $H_{drive} \sim \omega$ n-1 times) survive:

$$[\dots[H_{kin}, H_{drive}], H_{drive}, \dots, H_{drive}]$$

<u>but</u>: $H_{kin} \sim \partial_\theta^2$, 3- & higher nested commutators vanish identically ✓

- is there a more efficient way to compute this?
  → ideally, so we can go to order $1/\omega$

→ Kapitza pendulum in rotating frame

$$H(t) = \frac{p_\theta^2}{2m} - m\left(\omega_0^2 + A\,\omega\cos\omega t\right)\cos\theta$$

"problematic" scaling, messes up power counting!

<u>recall</u>:

$$H_{rot}(t) = V^\dagger(t)\, H(t)\, V(t) \underbrace{- i\, V^\dagger(t)\,\partial_t\, V(t)}$$

idea: use Galilean term to cancel $\propto \omega$ term by choosing $V(t)$ suitably

$$\text{e.g.}\quad V(t) = \exp\Big(-i \underbrace{\int_0^t (-mA\omega)\cos\omega t\, dt}_{=:\,\Delta(t)\,=\,-mA\sin\omega t} \times \cos\theta\Big)$$

<u>then</u>: $i V^\dagger \partial_t V = -mA\omega\cos\omega t\,\cos\theta = \omega f(t) H_{drive}$

compute:

$$V^\dagger(t) H(t) V(t) = e^{-i\Delta\cos\theta}\big(H_{kin} + H_{pot} + \omega f(t) H_{drive}\big) e^{+i\Delta\cos\theta}$$

$$= e^{-i\Delta\cos\theta} H_{kin}\, e^{+i\Delta\cos\theta} + H_{pot} + \omega f(t) H_{drive}$$

$$e^{-i\Delta\cos\theta}\frac{p^2}{2m} e^{+i\Delta\cos\theta} = \frac{p^2}{2m} + \frac{\Delta^2(t)}{2m}\sin^2\theta + \frac{\Delta(t)}{2m}\{\sin\theta, p\}_+$$

$\Rightarrow$

$$H_{rot}(t) = \frac{p^2}{2m} - m\omega_0^2\cos\theta + \frac{\Delta^2(t)}{2m}\sin^2\theta + \frac{\Delta(t)}{2m}\{\sin\theta, p\}_+$$

since $\Delta \propto O(\omega^0)$,

$$H_F^{(0)} = \frac{1}{T}\int_0^T dt\, H_{rot}(t)$$

$$= \frac{p^2}{2m} - m\omega_0^2\cos\theta + \frac{A^2 m}{4}\sin^2\theta \quad ✓$$

→ can easily identify the correct $1/\omega$ corrections

→ change of frame transformations lead to a resummation of IFE subseries

↳ non-perturbative effects

⇒ scaling drive amplitude w/ ω can lead to interesting phenomena in the time-averaged $H_F^{(0)}$

— Floquet engineering :

Q: how should we choose the drive Hamiltonian so we can prescribe the properties of H_F ?

Example 1 : dynamical stabilization

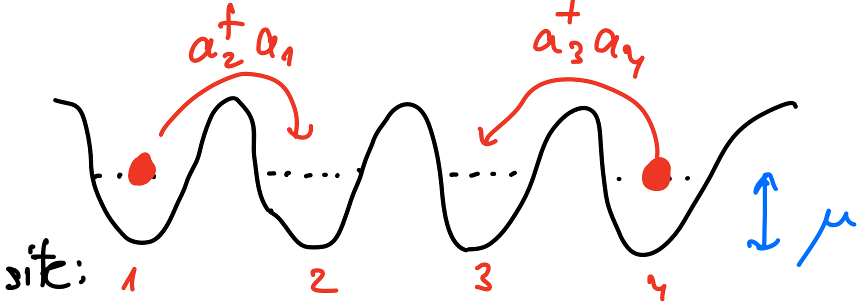
→ Kapitza oscillator (Piotr Kapitza, '51)

→ Paul trap (Wolfgang Paul, Nobel prize '89)

Example 2 : dynamical localization :

→ free fermion/boson chain :

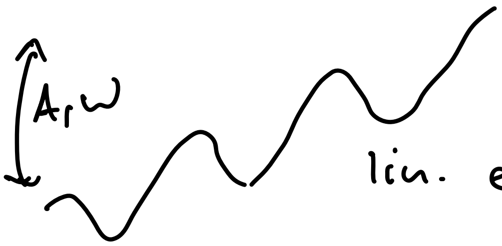
$$H_0 = \sum_j -J\left(a_{j+1}^\dagger a_j + h.c.\right) - \mu\, a_j^\dagger a_j$$



$n_j = a_j^\dagger a_j$ density on site j

hopping/tunneling delocalizes the wave fn (e.g. BEC, but also for fermions) want: to localize particles, i.e. suppress tunneling J

$$H_{drive}(t) = A\omega\cos\omega t \sum_j j\, n_j$$: oscillating electric potential



lattice + lin. electric pot. $\phi(r) \sim r$

total Hamiltonian:

eliminate by going to rot-frame

$$H_0 = \sum_j -J\left(a^\dagger_{j+1} a_j + h.c.\right) - \mu n_j + A\omega\cos\omega t\, j n_j$$

$$V(t) = e^{-iA\sin\omega t \sum_j j n_j} = \prod_j e^{-iA\sin\omega t\, j n_j}$$

$$H_{rot}(t) = \sum_j -J\, V^\dagger(t)\, a^\dagger_{j+1} a_j V(t) + h.c. - \mu n_j$$

$$= \sum_j -J\, V^\dagger a^\dagger_{j+1} V\, V^\dagger a_j V + h.c. - \mu n_j$$

$$= \sum_j -J\, e^{+iA\sin\omega t (j+1) n_{j+1}}\, a^\dagger_{j+1}\, e^{-iA\sin\omega t (j+1) n_{j+1}}$$

$$* \; e^{+iA\sin\omega t\, j n_j}\, a_j\, e^{-iA\sin\omega t\, j n_j} + h.c.$$

$$-\mu n_j$$

<u>need</u>: $e^{i\alpha n}\, a\, e^{-i\alpha n} =: F(\alpha)$ $(*)$; $\alpha := Aj\sin\omega t$

$$\partial_\alpha F = e^{i\alpha n}\underbrace{i[n,a]}_{=-ia}e^{-i\alpha n} = -iF$$

$$\Rightarrow F(\alpha) = F(0)\,e^{-i\alpha} \overset{(*)}{=} a e^{-i\alpha}$$

—

$$H_{rot}(t) = \sum_j -J\, e^{+iA\sin\omega t(j+1)} a^\dagger_{j+1} e^{-iA\sin\omega t\, j} a_j + h.c. - \mu n_j$$

$$= \sum_j -J\left(e^{+iA\sin\omega t} a^\dagger_{j+1} a_j + h.c.\right) - \mu n_j$$

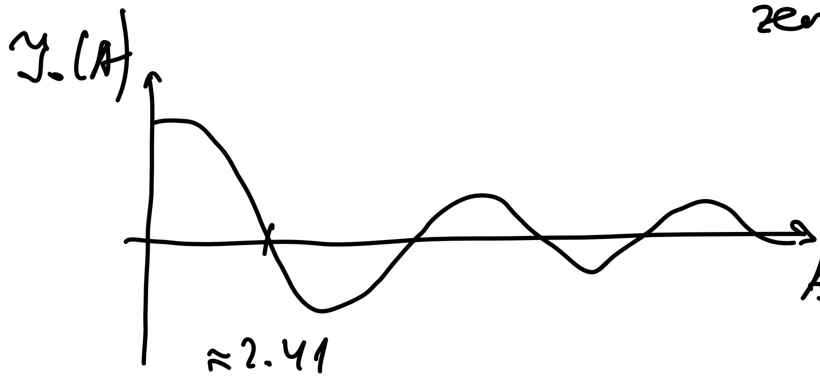
– apply IFE:

$$H_F^{(0)} = \frac{1}{T}\int_0^T dt\, H_{rot}(t)$$

$$= \sum_j -J_{eff}(A)\left(a^\dagger_{j+1}a_j + h.c.\right) - \mu n_j$$

where $J_{eff}(A) = \frac{1}{T}\int_0^T dt\, e^{-iA\sin\omega t} = \mathcal{J}_0(A)$

zeroth Bessel fn of 1^{st} kind



at $A \approx 2.41$

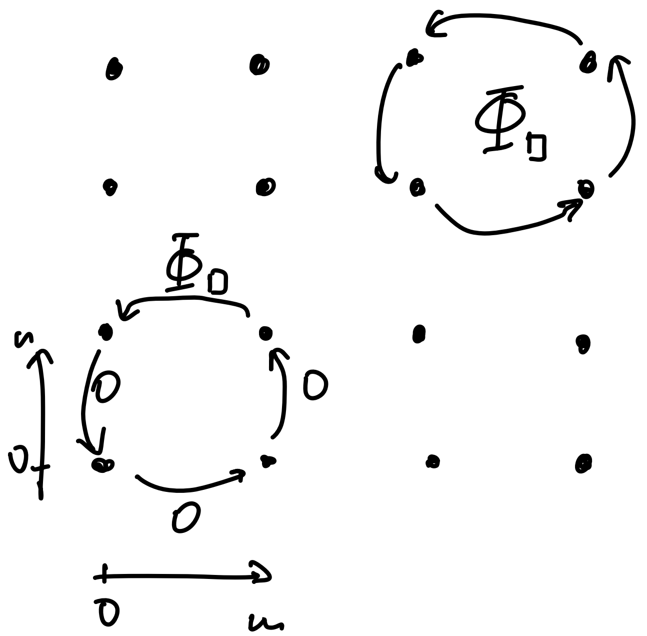
$J_{eff}(A) \approx 0 \rightarrow$ coherent hopping suppr.

<u>dynamical localization</u>

Example 3 : artificial magnetic fields

consider the Harper-Hofstadter Hamiltonian:

- 2D square lattice



$$H_{HH} = -K \sum_{m,n} e^{i\varphi_{mn}} a^\dagger_{m+1,n} a_{m,n} + h.c.$$

$$- J \sum_{m,n} a^\dagger_{m,n+1} a_{m,n} + h.c.$$

$$\varphi_{mn} = \Phi_\square (m+n)$$

$n_{mn} = a^\dagger_{mn} a_{mn}$: particle op.

- $\Phi_\square$: magnetic flux per plaquette
- magnetic field breaks time-reversal symmetry (i.e. cannot gauge away $e^{i\varphi_{mn}}$ term)

- in materials: $a^\dagger_{mn}$: electron operator
  → e^- are charged → couple to magnetic fields

  issue: $\Phi_\square$ limited by strength of magnetic field (→ technical challenge)

- quantum simulators
  - neutral "particles" → do not couple to EM fields!

idea: use Floquet engineering

$$H(t) = H_0 + H_{drive}(t)$$

$$H_0 = -\sum_{m,n} J_x\, a^\dagger_{m+1,n} a_{m,n} + J_y\, a^\dagger_{m,n+1} a_{m,n} + h.c.$$

gradient along x-only

$$H_{drive}(t) = \omega \sum_{m,n} \left[\frac{A}{2} \sin\left(\omega t - \varphi_{mn} + \frac{\Phi_\square}{2}\right) + m\right] n_{mn}$$

where $\varphi_{mn} = \Phi_\square (m+n)$ spatially inhom phase of drive

$\rightarrow$ breaks time-reversal

$\rightarrow$ go to rotating frame using

$$V(t) = e^{-i \int_0^t dt' H_{drive}(t')}$$ to eliminate ω-term

$$H_{rot}(t) = G(t) + G^\dagger(t)$$

$$G(t) = -\sum_{m,n} J_x\, e^{-i\zeta_\Phi \sin(\omega t - \varphi_{mn}) + i\omega t} a^\dagger_{m+1,n} a_{m,n} + h.c.$$

$$+ J_y\, e^{-i\zeta_\Phi \sin(\omega t - \varphi_{mn})} a^\dagger_{m,n+1} a_{m,n} + h.c.$$

$$\zeta_\Phi = A \sin(\Phi_\square/2)$$

- effective Hamiltonian:

use $$e^{i\alpha \sin(\omega t - \varphi)} = \sum_\ell \mathcal{J}_\ell(\alpha)\, e^{-i\ell(\omega t - \varphi)}$$

$$\frac{1}{T}\int_0^T dt\, e^{-i\zeta_\Phi \sin(\omega t - \varphi_{mn}) + i\omega t} =$$

$$= \sum_\ell \mathcal{J}_\ell(\zeta_\Phi) \underbrace{\int_0^T \frac{dt}{T}\, e^{-i\ell(\omega t - \varphi_{mn})} e^{i\omega t}}_{= e^{+i\varphi_{mn}} \delta_{\ell,1}} = \mathcal{J}_1(\zeta_\Phi)\, e^{i\varphi_{mn}}$$

$$H_F^{(0)} = -\sum_{m,n} \left(K e^{i\varphi_{mn}} a^\dagger_{m+1,n} a_{m,n} + h.c. \right) + \left(J a^\dagger_{m,n+1} a_{m,n} + h.c. \right)$$

where $K = J_x \mathcal{J}_1(\zeta_\Phi)$ & $J = J_y \mathcal{J}_0(\zeta_\Phi)$

$\rightarrow$ HH Hamiltonian

artificial magnetic field

- Example 4 : topological bands
  ($\hookrightarrow$ project)

- Floquet engineering is limited by
  1) laws of physics
  2) your own creativity !

- remarks :
  a) all of the above examples generalize to interacting systems (density - density interactions)
  b) Floquet systems are intrinsically out of equil.
  $[H(t_1), H(t_2)] \neq 0$
  $\Rightarrow$ energy conservation lost
  $\Rightarrow$ system can (and in generall will) absorb energy from drive $\rightsquigarrow$ heating
  - problematic for ordered states