Thm. (Gaston Floquet, 1883) (theory of ODE's)

let $H : \begin{array}{l} \mathbb{R} \longrightarrow \mathbb{C}^{n \times n} \\ t \longmapsto H(t) \end{array}$ be continuous, matrix-valued fn with period T: $H(t+T) = H(t)$ (Hamiltonian)

let $U(t)$ be the fundamental matrix (time-evo operator) to the first-order linear ODE:

$$i\partial_t \psi(t) = H(t)\,\psi(t) \quad ; \quad i\partial_t U(t) = H(t)U(t)$$

$$U(0) = \mathbb{1}$$

then:

1) $U(t+T)$ is also a fundamental matrix

2) there exists a non-singular, continuously diff'ble matrix-valued fn:

$$P : \begin{array}{l} \mathbb{R} \longrightarrow \mathbb{C}^{n \times n} \\ t \longmapsto P(t) \end{array} \quad \text{with period } T : \; P(t+T) = P(t)$$

and a time-independent matrix $H_F \in \mathbb{C}^{n \times n}$, s.t.

$$U(t) = P(t)\exp(-i\,t\,H_F)$$

$$U(T) = P(T)\,e^{-iTH_F} \underset{=}{\overset{U(0)=\mathbb{1}\;\Rightarrow\;P(0)=\mathbb{1}}{}} e^{-iTH_F}$$

Corollary: stroboscopically, i.e. at $t = \ell T$

$$\boxed{U(\ell T) = [U(T)]^{\ell} = e^{-i\ell T H_F}}$$

<u>Proof</u>: $i\partial_t U(t+T) = i\dot{U}(t+T) \underset{\text{def of } U}{=} H(t+T)\,U(t+T)$

1)

$= H(t)\,U(t+T)$ ✓

2) $U(t)$ & $U(t+T)$ are both fund. matrices

$\Rightarrow$ there is a <u>static linear transf.</u> that relates them:

$\Rightarrow U(t+T) = U(t)\,\underline{U_F}$ $(**)$

$U_F : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$

by existence of matrix log : $U_F = e^{-iTH_F}$

<u>set</u>: $P(t) := U(t)\,e^{+itH_F}$ $(*)$

check periodicity: $P(t+T) \overset{(*)}{=} U(t+T)\,e^{+i(t+T)H_F}$

$\overset{(**)}{=} U(t)\,\underbrace{U_F\,e^{+iTH_F}}_{=\mathbb{1}}\,e^{+itH_F} = U(t)e^{+itH_F}$

$\overset{(*)}{=} P(t)$ ✓ $P(t)$ periodic w/ period T

invert $(*)$: $U(t) = P(t)\,e^{-itH_F}$

<u>Rmks</u>:

1) note: thm requires a <u>linear</u> ODE

2) $H(t)$ need not be hermitian

but if $H(t) = H^\dagger(t) \Rightarrow H_F = H_F^\dagger$

- physical meaning: $U_{lab}(t,0) = P(t)\, \underbrace{e^{-itH_F}}_{= U_{rot}(t)}\, P^\dagger(0)$

→ looks like the transformation law for evo op's b/w lab & rot frames!

$P(t)$: rot $\longrightarrow$ lab

⇒ Hamiltonian in rot frame:

Floquet's thm

$$H_{rot}(t) = P^\dagger(t)\, H_{lab}(t)\, P(t) - i\, P^\dagger(t)\, \partial_t P(t) = H_F$$

time-indep.

- Floquet's thm: statement about existence of a ref. frame where dynamics of system is governed by the time-indep Hamiltonian H_F

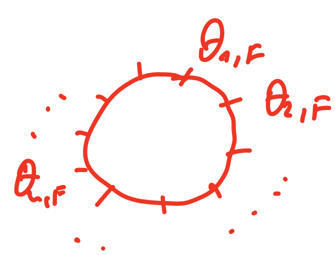
note: thm doesn't tell us how to compute H_F (how to find that special frame)!

- notation:
  - effective / Floquet Hamiltonian: H_F, H_{eff}
    → this is NOT the same as H_0
  - Floquet unitary: $U_F = U(T,0) = e^{-iTH_F}$
    → evo op. over one drive cycle / period T

→ eigendecomposition:

Floquet states

$$U_F = \sum_n e^{i\theta_{n,F}} |n_F\rangle\langle n_F|$$



spectrum of U_F: $\{\theta_{n,F}\}_n$ lives on unit circle

$\Rightarrow H_F = \sum_n \varepsilon_{n,F} |n_F\rangle\langle n_F| \quad , \quad T\varepsilon_{n,F} = \theta_{n,F}$

$\uparrow$ quasi-energies

q'energies defined mod ω : $e^{iT(\varepsilon_{n,F} + l\omega)} = e^{iT\varepsilon_{n,F}} \quad l \in \mathbb{Z}$

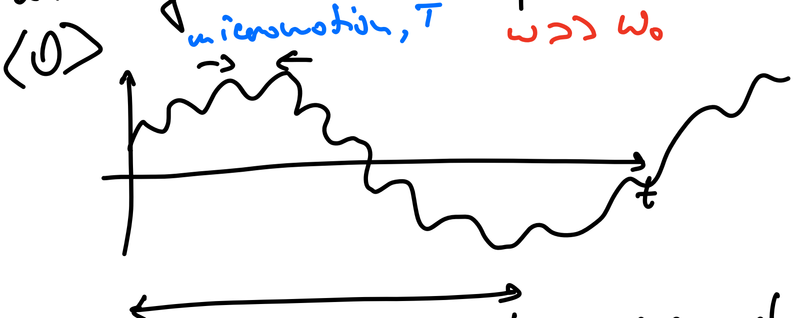
$\hookrightarrow$ analogy to quasi-momentum (Bloch's thm) in systems w/ spatial periodicity

- **kick operator**: $K(t)$: $\underbrace{P(t)}_{\text{micromotion operator}} = e^{-iK(t)}$

$\rightarrow K(t+T) = K(t)$

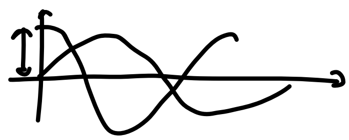
$\rightarrow$ at high drive freq's ω, $K(t)$ govern micromotion

$\omega \gg \omega_0$



long time scale, governed by H_F

- **Floquet gauge**: dependence on initial time t_0 (or phase of the drive)



$$U(t,t_0) = e^{-iK_F(t\,;\,t_0)}\, e^{-iH_F[t_0]}$$

$\rightarrow H_F[t_0]$ matrix depends on t_0

but $\varepsilon_{n,F}$ are indep. of t_0 ($\rightarrow$ hence t_0 is a gauge)

$\Rightarrow |n_F\rangle$ also depend on t_0

$\rightarrow$ kick op. $K_F(t\,;\,t_0)$ depends on t_0

How do we compute H_F & K_F in practice?

→ inverse-frequency expansions: $\omega \gg \omega_0$: $\omega \to \infty$ (or $T \to 0$)

1) simple case: step drives (similar for kicked drives)

$$U_F = e^{-i\frac{T}{2}H_1} e^{-i\frac{T}{2}H_0} \stackrel{!}{=} e^{-iTH_{FM}}$$

$$H_{FM} = \frac{i}{T}\log\left(e^{-i\frac{T}{2}H_1} e^{-i\frac{T}{2}H_0}\right)$$

$$= \frac{i}{T}\Bigg(-i\frac{T}{2}(H_0+H_1) + \left(\frac{-iT}{2}\right)^2 \frac{1}{2}[H_0,H_1]$$
$$+ \left(\frac{-iT}{2}\right)^3 \frac{1}{12}\Big([H_0,[H_0,H_1]] + (1\leftrightarrow 0)\Big)\Bigg)$$

$$= \frac{1}{2}(H_0+H_1) \quad = \mathcal{O}(T^0) := H_{FM}^{(0)}$$

$$-\frac{T}{8}\, i[H_0,H_1] := H_{FM}^{(1)} = \mathcal{O}(T^1)$$

$$-\frac{T^2}{96}\Big([H_0,[H_0,H_1]] + [H_1,[H_1,H_0]]\Big) := H_{FM}^{(2)} = \mathcal{O}(T^2)$$

$$\Rightarrow \quad H_{FM} = \sum_{n=0}^{\infty} H_{FM}^{(n)} \quad ; \quad H_{FM}^{(n)} \propto \frac{1}{\omega^n} \propto T^n$$

Baker - Campbell - Hausdorff (BCH) formula

2) generic time-periodic dependence:

Floquet - Magnus expansion:
(F M)

$$LHS = \mathcal{T}_t\, e^{-i\int_0^T dt\, H(t)} = U_F \overset{\text{Floquet}}{=} e^{-iTH_{FM}} = RHS$$

<u>assume:</u> $H_{FM} = \sum_{n=0}^{\infty} H_{FM}^{(n)} \sim \frac{1}{\omega^n}$ $\qquad H(t_j) = H_j$

$$LHS = \mathbb{1} - i\underbrace{\int_0^T dt_1\, H_1}_{\sim T} - \underbrace{\int_0^T dt_1 \int_0^{t_1} dt_2\, H_1 H_2}_{\sim T^2}$$

$$RHS = \mathbb{1} - iT\left(H_{FM}^{(0)} + H_{FM}^{(1)} + H_{FM}^{(2)} + \dots\right)$$
$$+ \frac{(iT)^2}{2}\left(H_{FM}^{(0)} + H_{FM}^{(1)} + \dots\right)\left(H_{FM}^{(0)} + H_{FM}^{(1)} + \dots\right)$$
$$+ \dots$$

$H_F^{(0)} = \frac{1}{T}\int_0^T dt\, H(t)$ time-averaged Hamiltonian

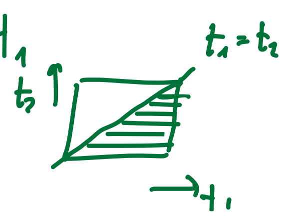
$$-\int_0^T dt_1 \int_0^{t_1} dt_2\, H_1 H_2 \overset{!}{=} -iT H_{FM}^{(1)} - \frac{T^2}{2}\left(H_{FM}^{(0)}\right)^2$$

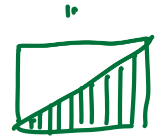
$$H_{FM}^{(1)} = \frac{i}{T}\left(\frac{\cancel{T^2}}{2}\left(\frac{1}{\cancel{T}}\int_0^T dt\, H(t)\right)^2 - \int_0^T dt_1 \int_0^{t_1} dt_2\, H_1 H_2\right)$$

$$= \frac{i}{T}\left(\frac{1}{2}\int_0^T dt_1 \int_0^T dt_2\, H_1 H_2 - \int_0^T dt_1 \int_0^{t_1} dt_2\, H_1 H_2\right)$$

$$= \frac{i}{T}\left(\frac{1}{2}\int_0^T dt_1 \left(\int_0^{t_1} dt_2 + \int_{t_1}^T dt_2\right) H_1 H_2 - 1\int_0^T dt_1 \int_0^{t_1} dt_2\, H_1 H_2\right)$$

$$= -\frac{i}{2T}\left(\int_0^T dt_1 \int_0^{t_1} dt_2\, H_1 H_2 - \underbrace{\int_0^T dt_1 \int_{t_1}^T dt_2\, H_1 H_2}_{\overset{1\leftrightarrow 2}{=} \int_0^T dt_2 \int_{t_2}^T dt_1\, H_2 H_1}\right)$$





$$= \int_0^T dt_1 \int_0^{t_1} dt_2 \; H_2 H_1$$

$$H_{FM}^{(1)} = -\frac{i}{2T} \int_0^T dt_1 \int_0^{t_1} dt_2 \; [H_1, H_2]$$

similarly:

$$H_{FM}^{(2)} = \frac{1}{3!\,T\,i^2} \int_0^T dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \; [H_1,[H_2,H_3]] + [H_3,[H_2,H_1]]$$

– determined H_{FM}, next comes K_{FM}:

$$\mathcal{T} e^{-i\int_0^t ds\, H(s)} = e^{-i K_{FM}(t)} e^{-it H_{FM}}$$

$$K_{FM}(t) = \sum_{n=0}^{\infty} \underbrace{K_{FM}^{(n)}(t)}_{\propto\, 1/\omega^n}$$

→ using expansion for $H_{FM}^{(n)}$:

$$K_{FM}^{(0)}(t) \equiv 0$$

$$K_{FM}^{(1)}(t) = \int_0^t ds \left(H(s) - H_{FM}^{(0)} \right) \sim O(1/\omega)$$

– Floquet-Magnus boundary condition

$$K_{FM}(\ell T) = 0 \quad , \ell \in \mathbb{Z}$$

3) van Vleck expansion:

$$U(t,0) = e^{-i K_{vV}(t;t_0)} e^{-it H_{vV}} e^{+i K_{vV}(t;t_0)}$$

indep. of Floquet gauge

– defining prop.'s:

(i) H_{vV} <u>indep.</u> of t_0 (all dependence is part of K_{vV})

(ii) $\int_0^T K_{vV}(t)\,dt = 0$

→ want to expand: $H_{vV} = \sum_n H_{vV}^{(n)}$

$K_{vV}(t) = \sum_n K_{vV}^{(n)}(t)$

<u>recall</u>: Floquet's thm defines a ref. frame where dynamics is generated by H_{vV}

<u>now</u>: construct ref. transf. order by order in $1/\omega^n$

↳ vV expansion

$$e^{iK} H e^{-iK} = H + i[K,H] - \frac{1}{2}[K,[K,H]] + \dots$$

$$-i\, e^{iK}\partial_t e^{-iK} = -\partial_t K - \frac{i}{2}[K, \partial_t K] + \frac{1}{6}[K,[K,\partial_t K]] + \dots$$

$$H_{vV} = e^{iK_{vV}(t)} H(t)\, e^{-iK_{vV}(t)} - i\, e^{iK_{vV}(t)}\partial_t e^{-iK_{vV}(t)}$$

$$= H(t) + i[K^{(1)}(t), H(t)] + \mathcal{O}(\omega^{-2})$$

$$- \frac{i}{2}[K^{(1)}(t), \partial_t K^{(1)}(t)] - \partial_t\left(K^{(1)}(t) + K^{(2)}(t)\right) + \mathcal{O}(\omega^{-2})$$

time-dep. on RHS has to vanish order by order in $1/\omega$ since LHS is time-indep.

expand: $H(t) = \sum_{\ell=-\infty}^{\infty} H_\ell\, e^{+i\ell\omega t}$

operator-valued Fourier coeff's!

$$= H_0 + \underbrace{\sum_{\ell\neq 0} H_\ell e^{i\ell\omega t} - \partial_t K^{(1)}(t)}_{\overset{!}{=} 0} - \partial_t K^{(2)}(t)$$

$$+ i\,[K^{(1)}(t), H(t)] - \frac{i}{2}[K^{(1)}(t), \partial_t K^{(1)}(t)]$$

$$\partial_t K = \sum_{\ell\neq 0} H_\ell e^{i\ell\omega t} \Rightarrow K(t) = \sum_{\ell\neq 0} \frac{1}{i\ell\omega} H_\ell e^{i\ell\omega t}$$

boundary condition

$\int_0^T K\,dt = 0 \Leftrightarrow K$ has no $\ell=0$ term

$$= H_0 + i\Big[\underbrace{\sum_{\ell\neq 0} \frac{1}{i\ell\omega} H_\ell e^{i\ell\omega t}}_{=K^{(1)}(t)}, \underbrace{\sum_m H_m e^{im\omega t}}_{=H(t)}\Big]$$

$$-\frac{i}{2}\Big[\underbrace{\sum_{\ell\neq 0} \frac{1}{i\ell\omega} H_\ell e^{i\ell\omega t}}_{=K^{(1)}(t)}, \underbrace{\sum_{m\neq 0} H_m e^{im\omega t}}_{=\partial_t K^{(1)}(t)}\Big]$$

$$-i\partial_t K^{(2)} + \mathcal{O}(\omega^{-2})$$

$$= H_0 + \frac{1}{\omega}\sum_{\ell\neq 0} \frac{1}{\ell} e^{i\ell\omega t}[H_\ell, H_0]$$

$$+ \frac{1}{2\omega}\sum_{\substack{\ell\\ m\neq 0}} \frac{1}{\ell} e^{i\ell\omega t}[H_\ell, H_m]$$

$$- \partial_t K^{(2)}(t) + \mathcal{O}(\omega^{-2})$$

$$= H_0 + \frac{1}{2\omega}\sum_{\ell\neq 0}\frac{1}{\ell}[H_\ell, H_{-\ell}]$$

$$\left.\begin{aligned}&+ \frac{1}{2\omega}\sum_{\ell\neq 0}\sum_{m\neq -\ell}\frac{1}{\ell} e^{-i(\ell+m)\omega t}[H_\ell, H_m]\\ &+ \frac{1}{\omega}\sum_{\ell\neq 0}\frac{1}{\ell} e^{i\ell\omega t}[H_\ell, H_0] - i\partial_t K^{(2)}(t)\end{aligned}\right] \overset{!}{=} 0$$

$$K_{vV}^{(1)} = \frac{1}{\omega} \sum_{\ell \neq 0} \frac{1}{i\ell} H_\ell \, e^{i\ell\omega t} = \dots = -\frac{1}{2} \int_0^{T+t} dt' \left(1 + 2\frac{t-t'}{T}\right) \bmod 2 \times H(t')$$

$$K^{(2)}(t) = \frac{1}{\omega^2} \sum_{\ell \neq 0} \frac{1}{i\ell^2} [H_\ell, H_0] \, e^{i\ell\omega t}$$

$$+ \frac{1}{2\omega^2} \sum_{\substack{\ell \neq 0 \\ m \neq -\ell}} \frac{1}{\ell(\ell+m)} [H_\ell, H_m] \, e^{i(\ell+m)\omega t}$$

$$H_{vV}^{(0)} = H_0 = \frac{1}{T} \int_0^T dt \; H(t)$$

$$H_{vV}^{(1)} = \frac{1}{2\omega} \sum_{\ell \neq 0} \frac{1}{\ell} [H_\ell, H_{-\ell}] = \dots = -\frac{1}{2} \int_0^T dt_1 \int_0^{t_1} dt_2 \left(1 - 2\frac{t_1 - t_2}{T}\right) \bmod 2 \times [H_1, H_2]$$