A) we discussed previously, we want
to solve (each of the) radial Schrödinger
equations

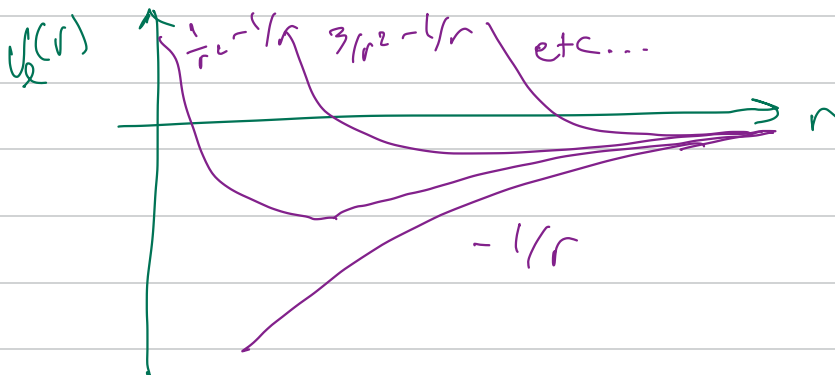$$\left(-\frac{1}{2} \frac{d^2}{dr^2} + \overbrace{\underbrace{\frac{l(l+1)}{2r^2} - \frac{1}{r}}_{} }^{V_l(r)} - E_{nl} \right) u_{nl}(r) = 0$$

to obtain the energy levels of hydrogen. This
involves solving the (in principle infinitely
many) 1D problems subject to boundary
conditions ~

$$u_{nl} \sim r^{l+1} \qquad , r \to 0$$
$$u_{nl} \sim 0 \qquad , r \to \infty \quad ~$$

with potential energies

$U_l(r)$ $\frac{1}{r^2} - \frac{1}{r}$ $3/r^2 - 1/r$ etc...

$-1/r$

We were gonna do this using a
slightly exotic method: SuSy Q-Mech.

Let us begin with the generic <u>1D</u> Ham:

$$H^{(1)} = -\frac{1}{2} d^2/dx^2 + V^{(1)}(x) \qquad \text{①}$$

where $V^{(1)}(x)$ has been shifted so that the ground state of $H^{(1)}$ sits at zero energy, i.e. $V^{(1)}(x) = V(x) - \mathcal{E}_0$.

This is a nice trick, since $\mathcal{U}_0(x)$, the ground state, is <u>nodeless</u> → so we have no pesky poles when trying to evaluate the following:

$$H^{(1)} \mathcal{U}_0 = 0$$
$$-\frac{\mathcal{U}_0''}{2} + V^{(1)} \mathcal{U}_0 = 0$$
$$\Rightarrow \underline{V^{(1)} = \frac{1}{2} \frac{\mathcal{U}_0''}{\mathcal{U}_0}} .$$

That's cool!!

What other weird ways can we write $V^{(1)}(x)$? How about –

$$V^{(1)}(x) = W(x)^2 - \frac{1}{\sqrt{2}} W'(x),$$

where $W(x)$ is the <u>SUPER</u> potential!

... But why?

It's simple - we can use $\hat{W}$ to define $\bar{A}$:
$$\hat{A} = \frac{1}{\sqrt{2}} \frac{d}{dx} + W(x)$$
$$\hat{A}^\dagger = -\frac{1}{\sqrt{2}} \frac{d}{dx} + W(x).$$
And these have the groovy property that:
$$H^{(1)} = A^\dagger A \quad !$$

How? Just check:
$$H^{(1)} \varphi = \left(-\frac{1}{\sqrt{2}} d_x + W\right)\left(\frac{1}{\sqrt{2}} d_x + W\right)\varphi$$
$$= -\frac{1}{2} \frac{d^2}{dx^2}\varphi - \frac{1}{\sqrt{2}}\left(W'\varphi + W\varphi'\right) + \frac{1}{\sqrt{2}} W\varphi' + W^2 \varphi$$
$$= \left(-\frac{1}{2} \frac{d^2}{dx^2} + \underbrace{W^2 - \frac{1}{\sqrt{2}} W'}\right)\varphi$$
this was how we defined $V^{(1)}$!

A has allowed us to <u>factor</u> any Ham!
But there's more! check out:
$$H^{(2)} = A A^\dagger.$$
$H^{(2)}$ has an eigenspectrum:

$$H^{(2)} \varphi_n = \mathcal{E}_n \varphi_n$$
$$\Rightarrow A A^\dagger \varphi_n = \mathcal{E}_n \varphi_n$$
$$\Rightarrow A^\dagger A \left(A^\dagger \varphi_n\right) = \mathcal{E}_n \left(A^\dagger \varphi_n\right)$$
$$\Rightarrow A^\dagger A \tilde{\varphi}_n = \mathcal{E}_n \tilde{\varphi}_n$$

Amazing!!
$H^{(1)}, H^{(2)}$ have
degenerate
spectra!!

Why does it matter? Notice that:

$$A A^\dagger \varphi = \left(\tfrac{1}{\sqrt{2}} d_x + W\right)\left(-\tfrac{1}{\sqrt{2}} d_x + W\right)\varphi$$

$$= -\tfrac{1}{2} \varphi'' + \tfrac{1}{\sqrt{2}}\left(W'\varphi + W\varphi'\right) - \tfrac{1}{\sqrt{2}} W \varphi' + W^2 \varphi$$

$$= -\tfrac{1}{2} \varphi'' + \underbrace{\left(\tfrac{W'}{\sqrt{2}} + W^2\right)}\varphi = -\tfrac{1}{2}\varphi'' + \underbrace{V^{(2)}}\varphi$$

And this is **not** the same as

$$V^{(1)} !$$

$$\to H^{(1)} = -d^2/dx^2 + V^{(1)} \quad \text{and} \quad H^{(2)} = -d^2/dx^2 + V^{(2)}$$

Little aside:

Since $V^{(1)}(x) = \tfrac{1}{2} \tfrac{\varphi_0''}{\varphi_0}$,

and $V^{(1)}(x) = -\tfrac{W'}{\sqrt{2}} + W^2(x)$,

we have $\divideontimes \quad \varphi_0''/\varphi_0 = -\sqrt{2}\, W'(x) + 2\, W(x)$.

hool trick: $\left(\varphi_0''/\varphi_0\right) = \left(\tfrac{\varphi_0'}{\varphi_0}\right)^2 + \left(\tfrac{\varphi_0'}{\varphi_0}\right)'$?

$$= \varphi_0'^2/\varphi_0^2 + \varphi_0''/\varphi_0 - \varphi_0'^2/\varphi_0^2$$

So: $\divideontimes$ is:

$$\left(\tfrac{\varphi_0'}{\varphi_0}\right)^2 + \left(\tfrac{\varphi_0'}{\varphi_0}\right)' = \left(-\sqrt{2}\, W(x)\right)' + \left(-\sqrt{2}\, W(x)\right)^2$$

So: $W(x) = -\tfrac{1}{\sqrt{2}}\, \varphi_0'/\varphi_0 = -\tfrac{1}{\sqrt{2}} \tfrac{d}{dx} \ln \varphi_0(x)$.

We can solve for all ε_n, φ_n of $V^{(1)}$, then use φ_0 to get W, then W to get $V^{(2)}$, and its spectrum for free! ! ! ! !

to the degenerate spectrum!

The one exception is the zero-energy state.
Notice that if $A^\dagger A \, \psi_0 = 0, \rightarrow A \psi_0 = 0.$
This is a 1st-order DE,

$$\frac{1}{\sqrt{2}} \psi_0' + W(x)\psi_0 = 0, \text{ with an easy solution!}$$

$$\rightarrow \psi_0^{(x)} = \exp\left[-\int_{-\infty}^{x} \sqrt{2} W(y) dy\right]$$

This can be normalized
   as long as $\int_{-\infty}^{x} W(y) dy$ goes to ∞ as $x \rightarrow \infty$.
So far, so good — so why can't $AA^\dagger \psi_0 = 0$ too?

This would require $\psi_0(x) = \exp\left[+\int_{-\infty}^{x} W(y) dy\right]$
                            now a +, not − !!

These cannot be simultaneously normalized!

PUNCHLINE:
   $H^{(1)} = A^\dagger A$ and $H^{(2)} = AA^\dagger$ have
The same spectrum even though they
correspond to very different potentials!

And: ψ_0 is always obtained from a simple
first order DE!

Let's do some examples. First, take
$$W(x) = ax^3.$$
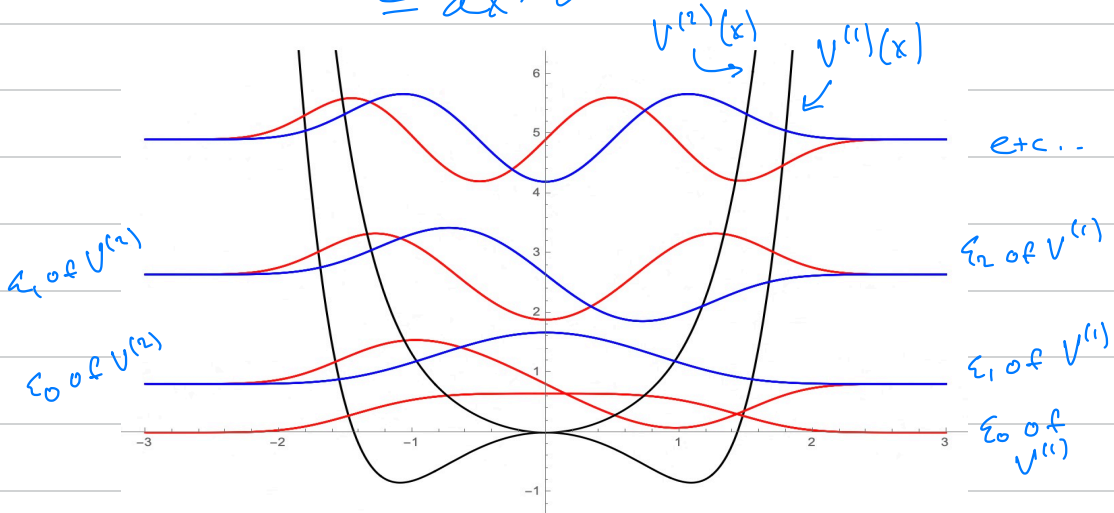
This gives:

- $V^{(1)}(x) = a^2x^6 - \frac{3}{v_2}ax^2$

- $V^{(2)}(x) = a^2x^6 - \frac{3}{v_2}ax^2$

- $\psi_0^{(x)} = N\exp\left[-\frac{v_2}{4}ay^4\Big|_{-\infty}^{x}\right]$

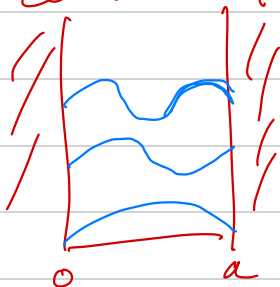$$= \underbrace{Ne^{-\infty}}_{\text{normalization}}\exp\left(-\frac{v_2}{4}ax^4\right)$$

- $W = -\frac{1}{v_2}\frac{\psi_0'}{\psi_0} = -\frac{1}{v_2}\left(-\frac{4}{4}ax^3 v_2\right)e^{-} \Big/ e^{-}$

$$= ax^3 \checkmark$$

$V^{(2)}(x) \quad V^{(1)}(x)$

etc..

ε_2 of $V^{(1)}$

ε_1 of $V^{(1)}$

ε_0 of $V^{(1)}$

ε_1 of $V^{(2)}$

ε_0 of $V^{(2)}$

This shows that two very different pots can have the same spectrum! But a little abstract...

For something a bit more down to earth,
consider the infinite square well.

this has solutions
$\sim \sin(kx)$ where
$$k = n\pi/a \rightarrow \mathcal{E} = \frac{k^2}{2} = \frac{n^2\pi^2}{2a^2}.$$

The SUPERPOT is thus:
$$W(x) = -\frac{1}{\sqrt{2}} \cdot \frac{\pi}{a} \cdot \cot\left(\frac{\pi x}{a}\right).$$

(set $a = \pi$ to make the rest easy..)

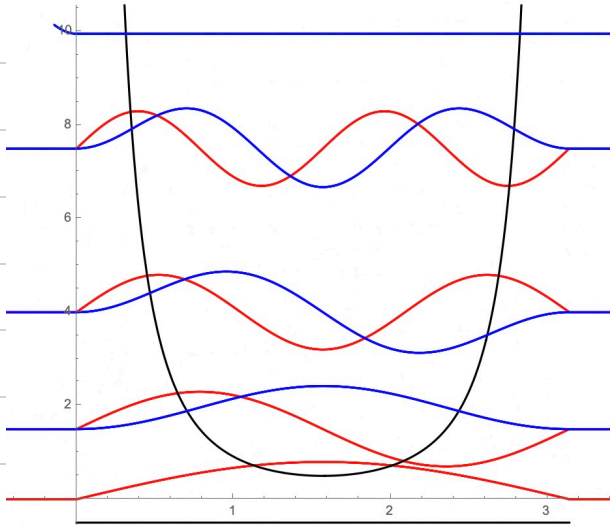And this gives 2 potentials...
$$U^{(1,2)}(x) = W^2 \pm \frac{1}{\sqrt{2}} W'(x)$$
$$= \frac{1}{2}\cot^2(x) \mp \frac{1}{2}\left(-\csc^2(x)\right)$$
$$= \frac{1}{2}\left(\frac{\cos^2(x) \mp 1}{\sin^2(x)}\right)$$
$$= -\frac{1}{2} \text{ or } \frac{1}{2}\left(2\csc^2(x) - 1\right).$$

$\rightarrow$ the first term gives the inf. sq. well, shifted
by $\mathcal{E}_0$. Cool! And the second is something
else – something weird – but something
w/ spectrum $\dfrac{n^2\pi^2}{2a^2}$, $n = 2, 3, \ldots$

Here we see
the states of
both systems...

This is a
good review article
if you want to
leeern more!

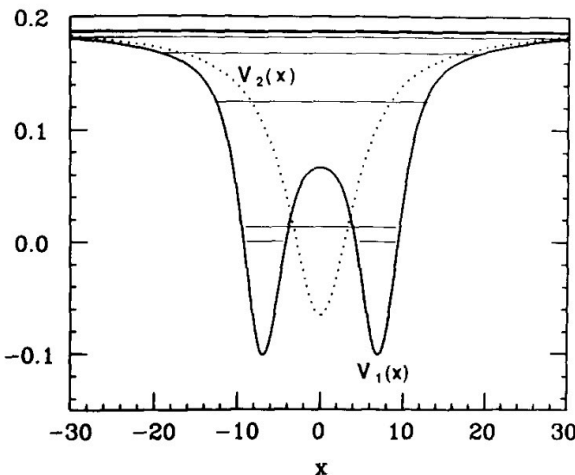# Supersymmetry and quantum mechanics

Fred Cooper[a], Avinash Khare[b], Uday Sukhatme[c]

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[b] Institute of Physics, Bhubaneswar 751005, India
[c] Department of Physics, University of Illinois at Chicago, Chicago, IL 60607, USA

e.g. how to solve
wachy susy systems
like this!

But now it's time to get back to our goal of solving the hydrogen atom:

$$H_e \Psi_e = \mathcal{E}_e \Psi_e,$$

where $H_e = -\frac{1}{2}\frac{d^2}{dr^2} + \frac{l(l+1)}{2r^2} - \frac{1}{r}.$

First off, let's figure out our SUPPORT!

From $\quad U^{(1)} = W^2 - \frac{1}{v_2}W' = \frac{l(l+1)}{2r^2} - \frac{1}{r} - \mathcal{E}_0$

we can guess the solution:

$$W(r) = C - D/r$$

$$\Rightarrow \frac{l(l+1)}{2r^2} - \frac{1}{r} - \mathcal{E}_0 = C^2 - \frac{2CD}{r} + \frac{D^2}{r^2} - \frac{1}{v_2}\frac{D}{r^2}$$

So: $\quad \underbrace{\mathcal{E}_0 = -C^2}_{A}, \quad \underbrace{-2CD = -1}_{B}, \quad \underbrace{D^2 - D/v_2 = \frac{l(l+1)}{2}}_{C}$

From C: $\quad D = \frac{l+1}{v_2}$

$\Rightarrow$ From B: $\quad C = \frac{1}{v_2(l+1)}$

$\rightarrow$ From A: $\quad \mathcal{E}_0 = -\frac{1}{2(l+1)^2}$

Wow! We already found the ground state energy for each l !!

As a sanity check, let's compute the wf:

$$\Psi_{0\ell}(r) = e^{-\int_0^r \sqrt{\frac{m}{\hbar^2}} W(y) \, dy}$$

$$= \exp\left[-\int_0^r \frac{1}{\ell+1} - \frac{(\ell+1)}{r} \, dr\right]$$

$$= \exp\left[-r/\ell+1 + (\ell+1) \ln r\right]$$

$$= r^{\ell+1} e^{-r/(\ell+1)}, \quad \text{up to normalization.}$$

Notice that the expected short- and long-range behavior holds, and this is a normalizable function! Hopefully, one we recognize from Kindergarten - QM!

Here's a nice way to visualize the gs energies we've obtained:

$$n = \frac{1}{\sqrt{-2E}}$$
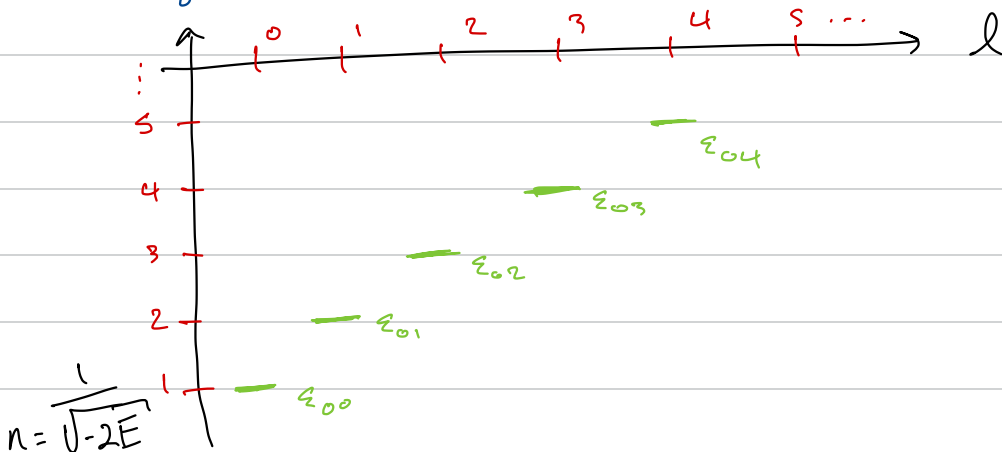
We've also determined

$$W(r) = \frac{1}{\sqrt{2}}\left[\frac{1}{\ell+1} - \frac{\ell+1}{r}\right]$$

With W, we can compute the A_ℓ's:

$$\hat{A}_\ell = \frac{1}{\sqrt{2}}\left(d/dr + \frac{1}{\ell+1} - \frac{\ell+1}{r}\right)$$

$$\hat{A}_\ell^\dagger = \frac{1}{\sqrt{2}}\left(-d/dr + \frac{1}{\ell+1} - \frac{\ell+1}{r}\right)$$

These give us the Hams:

- $H_\ell^{(1)} = A_\ell^\dagger A_\ell = \frac{1}{2}\left(-d^2/dr^2 + \underbrace{\frac{(\ell+1)\ell}{r^2}}_{\ell\text{-dependence is normal}} - \frac{2}{r} + \frac{1}{(\ell+1)^2}\right)$

$$\underbrace{\phantom{-d^2/dr^2 + \frac{(\ell+1)\ell}{r^2} - \frac{2}{r}}}_{\text{this is } H_\ell!}$$

$$= H_\ell^{coulomb} - \underbrace{\left(-\frac{1}{2(\ell+1)^2}\right)}_{\varepsilon_{0\ell}} \quad \textcircled{2}$$

- $H_\ell^{(2)} = A_\ell A_\ell^\dagger = \frac{1}{2}\left(-d^2/dr^2 + \overbrace{\frac{(\ell+1)(\ell+2)}{r^2}}^{\ell\text{-dep. is of }\ell+1} - \frac{2}{r} + \frac{1}{(\ell+1)^2}\right)$

$$= H_{\ell+1}^{coulomb} + \frac{1}{2(\ell+1)^2}. \quad \textcircled{1}$$

Sanity check #2: $H_\ell^{(1)}$ is the correct Ham, consisting of the Coulomb radial Ham less the ground state energy.

Notice that ⓪ leads us to:

$$H_{\ell+1}^{(1)} = H_{\ell+1}^{coulomb} - \left(-\frac{1}{2(\ell+2)^2} \right)$$

$$\rightarrow H_{\ell+1}^{coul} = H_{\ell+1}^{(1)} - \frac{1}{2(\ell+2)^2} \qquad ②$$

From ① and ② together, we have:

$$H_\ell^{(2)} = H_{\ell+1}^{(1)} + \frac{1}{2(\ell+1)^2} - \frac{1}{2(\ell+2)^2} \qquad ③$$

Now we want to use these to generate the (as yet unknown) excited states.

• Let ③ act on the g.s. of $H_{\ell+1}^{(1)}$:

$$H_\ell^{(2)} \phi_{0,\ell+1}^{(1)} = \underbrace{\left(H_{\ell+1}^{(1)} + \frac{1}{2(\ell+1)^2} - \frac{1}{2(\ell+2)^2} \right)}_{=0, \ b/c \ it \ is \ the \ gs.} \underbrace{\phi_{0,\ell+1}^{(1)}}$$

$$\rightarrow H_\ell^{(2)} \phi_{0,\ell+1}^{(1)} = \underbrace{\left(0 + \frac{1}{2(\ell+1)^2} - \frac{1}{2(\ell+2)^2} \right)}_{\not\Rightarrow} \phi_{0,\ell+1}^{(c)} \circ$$

So: ☆ is an eval of $H_\ell^{(2)}$.

So far, so good!!

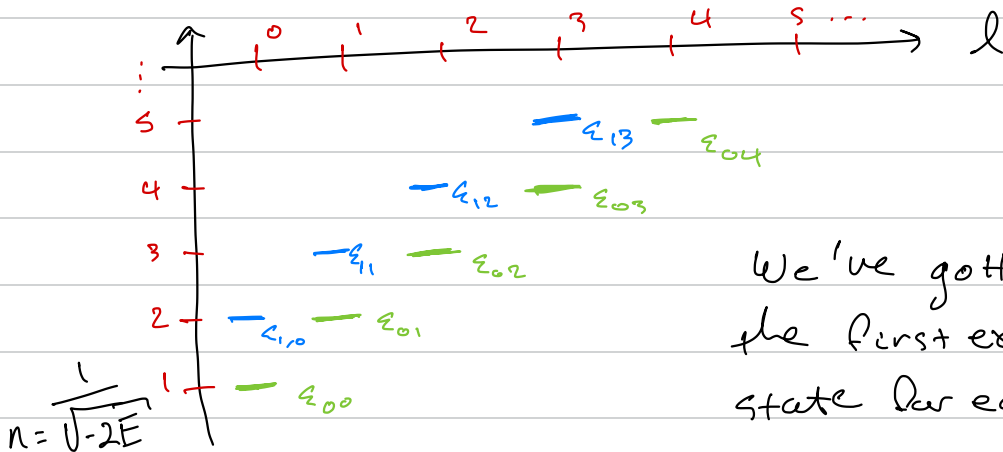But by SUSY, non-zero evals of $H_\ell^{(2)}$
are shared by $H_\ell^{(1)}$!

$\rightarrow \quad A_\ell^+ H_\ell^{(2)} \phi_{0,\ell+1}^{(1)} = \cancel{\bigstar} A_\ell^+ \phi_{0,\ell+1}^{(1)}$

$\rightarrow \quad \underbrace{A_\ell^+ A_\ell}_{H_\ell^{(1)}} \underbrace{\left(A_\ell^+ \phi_{0,\ell+1}^{(1)} \right)}_{\phi_{1,\ell}^{(1)}} = \underbrace{\cancel{\bigstar}}_{\cancel{\bigstar}} \underbrace{\left(A_\ell^+ \phi_{0,\ell+1}^{(1)} \right)}_{\phi_{1,\ell}^{(1)}}$

So: $\quad \bigstar = \dfrac{1}{2(\ell+1)^2} - \dfrac{1}{2(\ell+2)^2} \quad$ is the

corresponding eigenvalue of state $\phi_{1,\ell}^{(1)}$!

And thus, from ⑥: $\quad \mathcal{E}_{1,\ell}^{\text{coulomb}} = -\dfrac{1}{2(\ell+2)^2}$

We've gotten
the first excited
state for each ℓ!

$n = \dfrac{1}{\sqrt{-2E}}$

Let's go around the loop one more time!
We act on the first excited state of $H^{(1)}_{\ell+1}$
with $H^{(2)}_{\ell}$:

$$H^{(2)}_{\ell}\, \phi^{(1)}_{1,\ell+1} = \left(\underbrace{H^{(1)}_{\ell+1}}_{} + \underbrace{\frac{1}{2(\ell+1)^2} - \frac{1}{2(\ell+2)^2}}_{\frac{1}{2(\ell+2)^2} - \frac{1}{2(\ell+3)^2}, \text{ see page above}} \right) \underbrace{\phi^{(1)}_{1,\ell+1}}_{}$$

$$= \frac{1}{2}\left(\frac{1}{(\ell+2)^2} - \frac{1}{(\ell+3)^2} + \frac{1}{(\ell+1)^2} - \frac{1}{(\ell+2)^2} \right) \phi^{(1)}_{1,\ell+1}$$

So: $\quad H^{(2)}_{\ell}\, \phi^{(1)}_{1,\ell+1} = A_{\ell} A^{+}_{\ell}\, \phi^{(1)}_{1,\ell+1} = \left[\frac{1}{2(\ell+1)^2} - \frac{1}{2(\ell+3)^2} \right] \phi^{(1)}_{1,\ell+1}$

$\rightarrow$ multiply by A^{+}_{ℓ} ...

$\rightarrow \quad H^{(1)}_{\ell} \underbrace{\left[A^{+}_{\ell}\, \phi^{(1)}_{1,\ell+1} \right]}_{\phi^{(1)}_{2,\ell}} = \left[\frac{1}{2(\ell+1)^2} - \frac{1}{2(\ell+3)^2} \right] \phi^{(1)}_{2,\ell}$

Subtracting the offset energy gives:

$$H^{\text{coulomb}}_{\ell}\, \phi^{(1)}_{2,\ell} = -\frac{1}{2(\ell+3)^2}\, \phi^{(1)}_{2,\ell}$$

Looks like we have a pattern!

$$\mathcal{E}_{n,\ell} = -\frac{1}{2(n+\ell+1)^2}$$

$n = \dfrac{1}{\sqrt{-2E}}$

Energy level diagram with ℓ axis (0, 1, 2, 3, 4, 5 ...) horizontal and n axis (1, 2, 3, 4, 5, ...) vertical, showing states $\mathcal{E}_{00}$, $\mathcal{E}_{10}$, $\mathcal{E}_{01}$, $\mathcal{E}_{20}$, $\mathcal{E}_{11}$, $\mathcal{E}_{02}$, $\mathcal{E}_{30}$, $\mathcal{E}_{21}$, $\mathcal{E}_{12}$, $\mathcal{E}_{03}$, $\mathcal{E}_{31}$, $\mathcal{E}_{22}$, $\mathcal{E}_{13}$, $\mathcal{E}_{04}$.
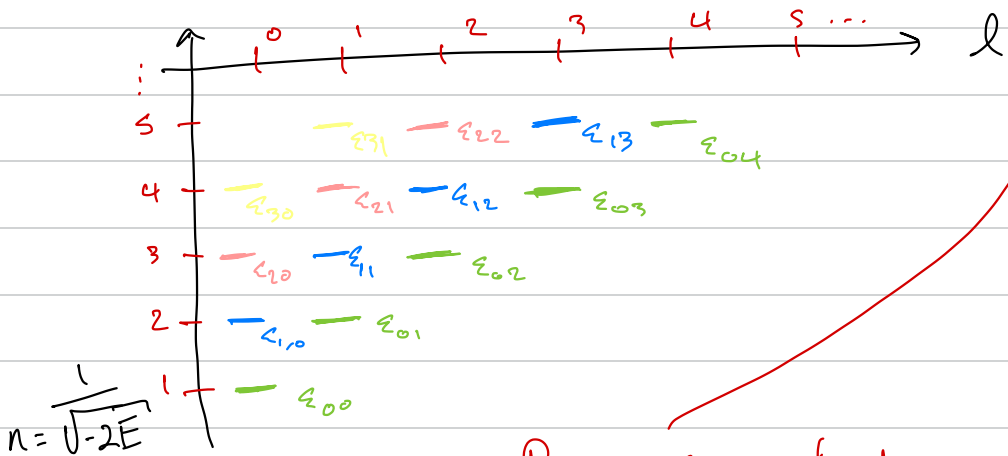
Recognize that we can relabel $n + \ell \to N$, an integer, and hence recover the same Rydberg formula as before.

Second diagram with axis $\ell = 0, 1, 2, 3, 4$ horizontal and $\dfrac{1}{\sqrt{-2\mathcal{E}}}$ labeled vertical (1, 2, 3, 4, 5), showing ϕ_{00}, ϕ_{10}, ϕ_{01}, ϕ_{11}, ϕ_{02}, ϕ_{12}, ϕ_{03}, with ladder operators A_0, A_1, A_2, A_0^+, A_1^+, A_2^+. "etc.."
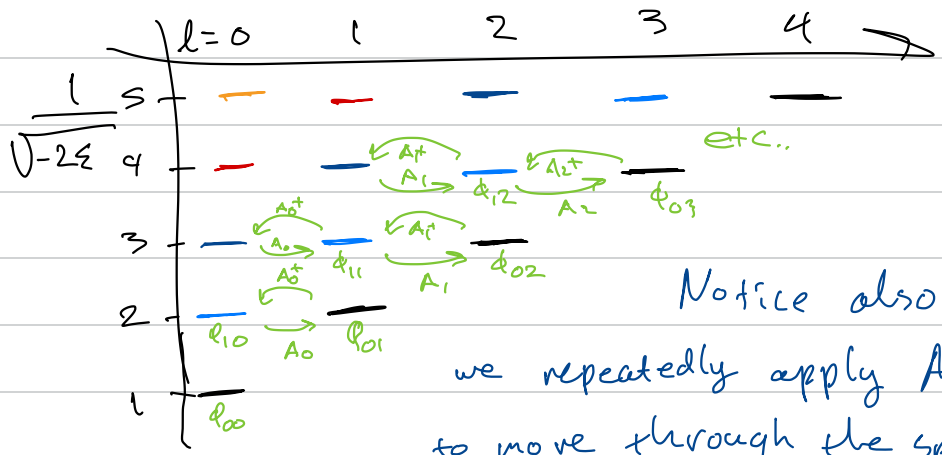
Notice also how we repeatedly apply A or A^+ to move through the spectrum!

This procedure we just did is part of a
much larger context of 'shape invariant
potentials', a category of 1D potentials
defined by the relationship

$$V^{(2)}(x, a_1) = V^{(1)}(x, a_2) + R(a_1)$$

↙ indep. or x

$$a_2 = f(a_1).$$

This lets us define a family of Hams

$$H_s = -\frac{1}{2} \frac{d^2}{dx^2} + V_1(x, a_s) + \sum_{k=1}^{s-1} R(a_k)$$

$$a_s = f(f(f \cdots (a_1))) |_{\cdots s-1}$$

Who cares? Well,

$$H_{s+1} = -\frac{1}{2} \frac{d^2}{dx^2} + \underbrace{V_1(x, a_{s+1})}_{= V_2(x, a_s) - R(a_s)} + \sum_{k=1}^{s} R(a_k)$$

$$\rightarrow = -\frac{1}{2} \frac{d^2}{dx^2} + V_2(x, a_s) + \sum_{k=1}^{s-1} R(a_k)$$

H_s and H_{s+1} are SUSY HAMS → so
their spectra are identical except for H_s's
GS, w/. $E_0^s = \sum_{k=1}^{s-1} R(a_k)$.

Doing this for all s values gives the spectrum

$$\mathcal{E}_n = \sum_{k=1}^{n} R(a_k), \quad \mathcal{E}_0 = 0.$$

Check all this w/ Mr. Coulomb!

In conclusion, we used SUSY HAMS to solve the hydrogen problem. This introduced a wider concept of factorizable Hamiltonians and SIPs, which are another way of categorizing all analytically solvable systems!
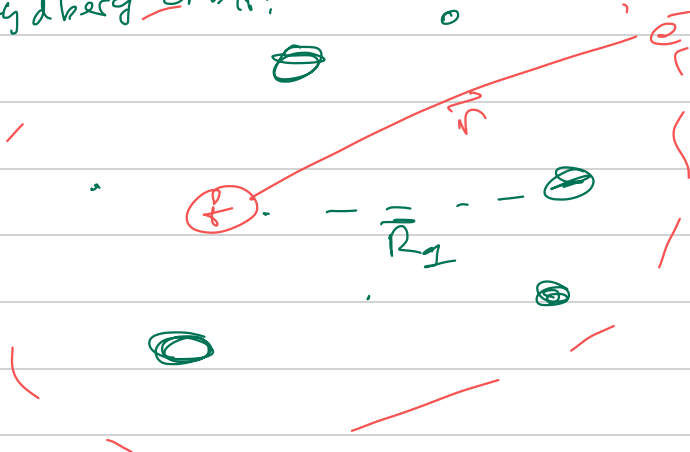
Historically, Mr. Schrö himself was the first to factorize the Coulomb problem!

For more reading:
Cooper, Khare, & Sukhatme Phy. Rep. <u>251</u>, 267 (1995)

Before closing the book on SUSY HAMs, here's one more connection to Rydberg physics. Consider a Rydberg atom in a dense gas, where many ground state atoms lie inside the Rydberg orbit:

The interaction of the Ryd e^- w/ each gs atom can be very well approx'd by

$$U(\vec{r}, \vec{R_g}) = 2\pi a \, \delta^3(\vec{r} - \vec{R_g})$$

where a = the zero energy scattering length.

This was first studied by Fermi in 1934 (we will say more about this later)

To make a long story short: we want to see how these gs atoms perturb the Ryd atom with principal QN n.

The l states are degenerate, so we must
diagonalize $\hat{V} = 2\pi a \sum\limits_{q=1}^{N} \delta(\vec{r} - \vec{R}_q)$

$$\rightarrow V_{nlm, nl'm'} = 2\pi a \sum\limits_{q=1}^{N} \int \phi^*_{nlm}(\vec{r}) \delta^3(\vec{r} - \vec{R}) \phi_{nl'm'}(\vec{r})$$

$$= 2\pi a \sum\limits_{q=1}^{N} \phi^*_{nlm}(R_q) \phi_{nl'm'}(R_q).$$

This is an $n^2 \times n^2$ matrix - yikes to diagonalize!

<u>But wait!</u> Define: $A^\dagger = \overrightarrow{\phi_{nlm}(R_q)}$

$\quad\quad\quad\quad\quad\quad\quad\quad A = \phi_{nlm}(R_q)$

These are 2 rectangular matrices...

All that SUSY stuff says:
$$\underline{V} = 2\pi a \, A^\dagger A$$
$$\rightarrow \quad \underline{V}\vec{c} = \varepsilon \vec{c} = A^\dagger A \vec{c}$$
$$\rightarrow \quad \varepsilon(A\vec{c}) = \underline{AA^\dagger}(A\vec{c})$$
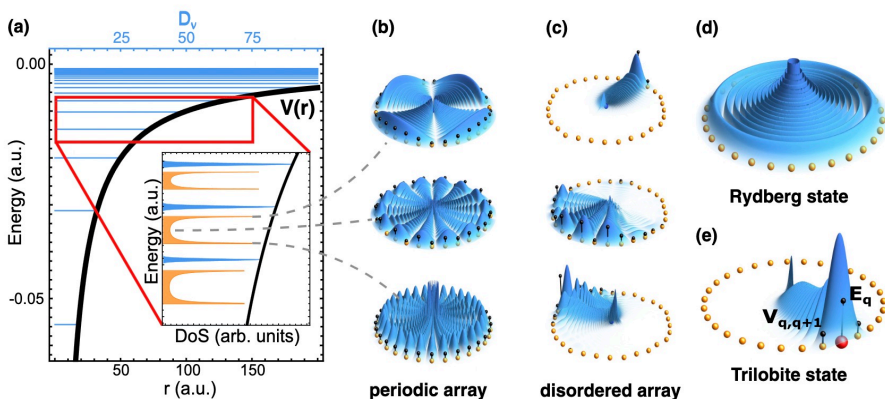we only need this $N \times N$ matrix!

Not only is this numerically way easier, it is conceptually very cool as it links two SUSY-like flavs!

$$H_1 = -\frac{d^2}{dr^2} + \frac{l(l+1)}{2r^2} - \frac{1}{r} + 2\pi a \sum_{q=1}^{N} \delta^3(\vec{r}-\vec{R}_q)$$

(a perturbed Rydberg atom)

$$H_2 = \sum_{q=1}^{N} E_q |q\rangle\langle q| + \sum_{q,q'}^{N} V_{qq'} |q\rangle\langle q'|$$

(a tight-binding lattice!)

(a)   (b)   (c)   (d)

Rydberg state

(e)

E_q

$V_{q,q+1}$

Trilobite state

periodic array   disordered array

See MTE, Eisfeld, and Rost Phys. Rev. Research **5**, 033032 (2023)

Anderson localization of a Rydberg e^-!

One more "application":

U. Kostelecky + M.M. Nieto,
"Evidence for a Phenomenological
Supersymmetry in Atomic Physics"
PRL 53 2285 (1984).

The authors use the $U^{(1)}$ and $U^{(2)}$ we
derived for the Coulomb Hamiltonian,
w/ $U^{(2)}$ having the same spectrum as $U^{(1)}$,
minus the gs.

But then they say that this is like an
atom w/o the 1s orbitals, i.e. Li:

Li: $1s^2 \underline{2s}$,
the spectrum of these states
should resemble those of H!
And, the same thing goes for Na
Na: $1s^2 2s^2 3s \ldots$

They claim in the end that similarities
between the excited state energies of
different atomic species imply a kind
of evidence for supersymmetry!

Can this possibly be true? Stay tuned!

In conclusion, in this lecture we learned:

- How to **factorize** a Hamiltonian
  $$H = A^\dagger A$$
  so that its SuSy-partner
  $$\tilde{H} = AA^\dagger$$
  could be found
- That H and $\tilde{H}$ have degenerate spectra except for the ground state.
- How to use this to:
  - relate one H w/ another to solve 2 problems for the price of 1.
  - Build up the solution of the Coulomb problem directly.

Although this SuSy stuff won't be too useful in general applications, we can actually use it now to introduce one major topic of this course,
Quantum Defect Theory!

This example comes from:

U. Kostelecky & M.M. Nieto,
"Evidence for a Phenomenological
Supersymmetry in Atomic Physics"
PRL 53 2285 (1984)

These authors start from the SUSY partner
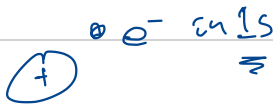potentials that we used,

$$V^{(1)} = -\frac{1}{r} + \frac{\ell(\ell+1)}{2r^2} + \frac{1}{2(\ell+1)^2}$$
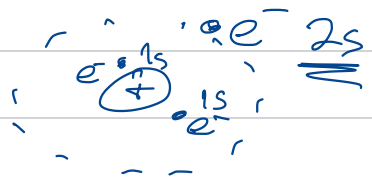$$V^{(2)} = -\frac{1}{r} + \frac{(\ell+1)(\ell+2)}{2r^2} + \frac{1}{2(\ell+1)^2},$$

and then claim: since $H^{(2)} = -\frac{1}{2}\frac{d^2}{dr^2} + V^{(2)}$ has
the same spectrum as $H^{(1)} = -\frac{1}{2}\frac{d^2}{dr^2} + V^{(1)}$ but
with the ground state removed, then $H^{(2)}$
describes a system with the 1s orbital removed.

So: in the absence of electron-electron
interactions, $H^{(2)}$ is the Hamiltonian of Li!

H: Li:

⊕ e⁻ in 1s   r ⌢ ⊙e⁻ 2s

(+)   e⁻•1s
                            (+) 1s
                              •e⁻

The authors then go on to claim that this
means that atomic spectra contain evidence
for supersymmetry! (And that this supersymmetry
is broken by e-e interactions.)

How do they justify this claim? With
experiment, of course!

TABLE I. Energy differences (in units of 10^4 cm^{-1}) between selected levels of the
alkali-metal atoms (Ref. 6). The best agreements with the H levels are underlined and
are those that one would expect from the discussion in the text. → those where e-e

int
is
yeah.

	H	Li			Na		
n	$s/p/d$	s	p	d	s	p	d
8-9	0.0360	0.0416	0.0366	<u>0.0360</u>	0.0607	0.0496	<u>0.0362</u>
6-7	0.0808	0.0980	0.0826	<u>0.0810</u>	0.164	0.124	<u>0.0814</u>
4-5	0.245	0.329	0.255	<u>0.247</u>	0.746	0.477	<u>0.249</u>
3-4	0.533	0.781	0.555	<u>0.534</u>	2.57	1.33	<u>0.538</u>
2-3	1.52	2.72	<u>1.60</u>	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Although agreement between the underlined
transitions seems pretty good, you might
find this line of argument a bit fishy...
and you should!
To see why, we'll need to solve the
Coulomb problem a few more times...